Multirate Digital Signal Processing

Sampling Rate Conversion

If a digital signal is formed by properly sampling an analog signal, it is possible in principle, to convert it back to analog form and re-sample it at a new sampling rate. But it would be better to directly change from one sampling rate to another without going through the analog form.

If a signal $\mathrm{x}[n]$ is formed by properly sampling $\mathrm{x}(t)$ then

$$\mathrm{x}(t) = 2\left(f_c / f_{sx}\right)\sum_{n=-\infty}^{\infty} \mathrm{x}[n]\,\mathrm{sinc}\left(2 f_c\left(t - nT_{sx}\right)\right)$$

where f_c is the corner frequency of a filter with impulse response

$$\mathrm{h}(t) = 2 f_c T_{sx}\,\mathrm{sinc}\left(2 f_c t\right) \xleftrightarrow{\mathcal{F}} \mathrm{H}(f) = T_{sx}\begin{cases} 1 \ , \ 0 < |f| < f_c \\ 0 \ , \ \text{otherwise} \end{cases}$$

Sampling Rate Conversion

$$\mathrm{x}(t) = 2(f_c / f_{sx}) \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_{sx}) \operatorname{sinc}(2 f_c (t - nT_{sx}))$$

If we sample $\mathrm{x}(t)$ at a rate $f_{sy} > f_{s\mathrm{x}}$ to form $\mathrm{y}[n]$ then

$$\mathrm{y}[m] = \mathrm{x}(mT_{sy}) = \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_{sx}) \operatorname{sinc}(f_{sx}(mT_{sy} - nT_{sx}))$$

In the special case in which $T_{sx} = T_{sy}$,

$$\mathrm{y}[m] = \mathrm{x}(mT_{sx}) = \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_{sx}) \operatorname{sinc}(f_{sx}(m-n)T_{sx}) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] \operatorname{sinc}(m-n)$$

which is a convolution sum $\mathrm{x}[n] * \operatorname{sinc}(n)$ and since $\operatorname{sinc}(n) = \delta[n]$

$$\mathrm{y}[m] = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\delta[m-n] = \mathrm{x}[m] * \delta[m] = \mathrm{x}[m]$$

as expected.

Sampling Rate Conversion

In the general case $T_{sx} \neq T_{sy}$,

$$y\left[m\right] = \sum_{n=-\infty}^{\infty} x\left(nT_{sx}\right)\operatorname{sinc}\left(f_{sx}\left(mT_{sy} - nT_{sx}\right)\right) \ , \ f_{sy} > f_{sx}$$

or

$$y\left[m\right] = \sum_{n=-\infty}^{\infty} x\left(nT_{sx}\right)\operatorname{sinc}\left(mT_{sy} / T_{sx} - n\right) \ , \ f_{sy} > f_{sx}$$

Now let $mT_{sy} / T_{sx} = k_m + \Delta_m$ where $k_m = \left\lfloor mT_{sy} / T_{sx} \right\rfloor$

and $\Delta_m = mT_{sx} / T_{sy} - k_m$. Then

$$y\left[m\right] = \sum_{n=-\infty}^{\infty} x\left(nT_{sx}\right)\operatorname{sinc}\left(k_m + \Delta_m - n\right) \ , \ f_{sy} > f_{sx}$$

Sampling Rate Conversion

Letting $k = k_m - n$,

$$y[m] = \sum_{k=-\infty}^{\infty} x\left(\left(k_m - k\right)T_{sx}\right)\operatorname{sinc}\left(k + \Delta_m\right) \ , \ f_{sy} > f_{sx}$$

This general process for sampling-rate conversion is a **time-variant** convolution sum. If the conversion factor is a ratio of integers (a rational number), the process can be considerably simpler.

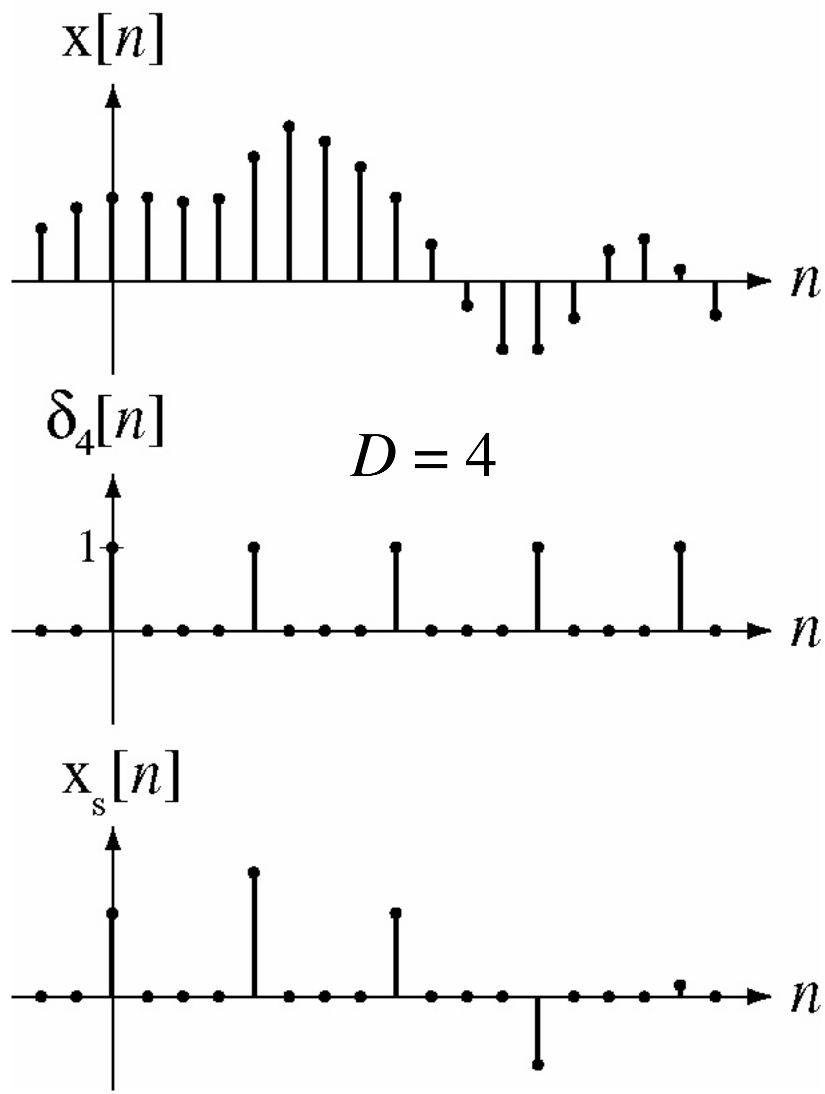
Sampling Rate Conversion

Downsampling by a factor D

Sample a signal $\mathrm{x}[n]$ by multiplying it by a periodic impulse

$$\delta_D[n] = \sum_{m=-\infty}^{\infty} \delta[n - mD]$$

to produce $\mathrm{x}_s[n] = \mathrm{x}[n]\delta_D[n]$.



Sampling Rate Conversion

The discrete-time Fourier series harmonic function for $\delta_D[n]$ is

$$\Delta_D[k] = \frac{1}{D}\sum_{n=\langle D\rangle}\left(\sum_{m=-\infty}^{\infty}\delta[n-mD]\right)e^{-j2\pi kn/D}$$

$$= \frac{1}{D}\sum_{n=0}^{D-1}e^{-j2\pi kn/D}\sum_{m=-\infty}^{\infty}\delta[n-mD] = 1/D$$

Therefore

$$\delta_D[n] = \sum_{k=0}^{D-1}\Delta_D[k]e^{j2\pi kn/D} = \frac{1}{D}\sum_{k=0}^{D-1}e^{j2\pi kn/D}$$

The z transform of $\mathrm{x}_s[n] = \mathrm{x}[n]\delta_D[n]$ is

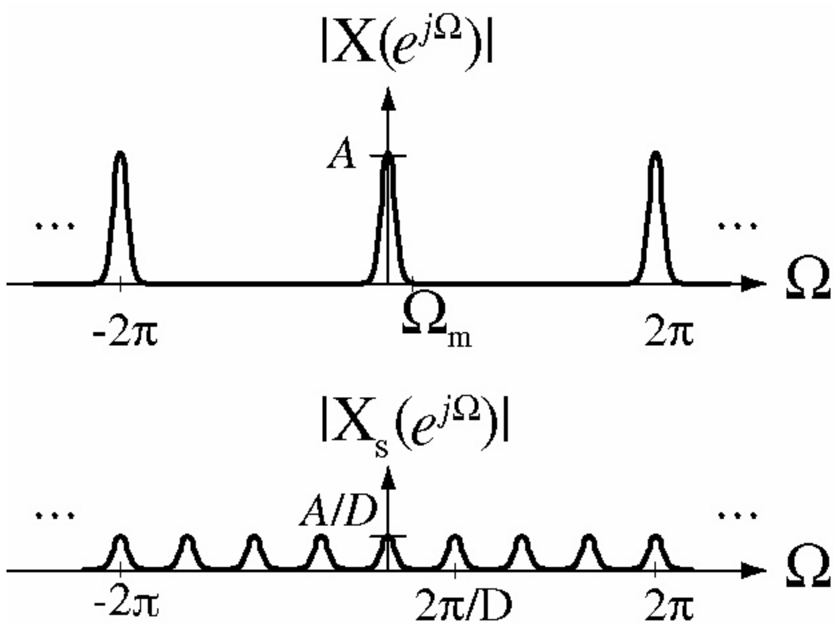
$$\mathrm{X}_s(z) = \sum_{n=-\infty}^{\infty}\mathrm{x}[n]\delta_D[n]z^{-n} = \frac{1}{D}\sum_{n=-\infty}^{\infty}\mathrm{x}[n]\left(\sum_{k=0}^{D-1}e^{j2\pi kn/D}\right)z^{-n}$$

Sampling Rate Conversion

$$X_s(z) = \frac{1}{D}\sum_{k=0}^{D-1}\sum_{n=-\infty}^{\infty} x[n] e^{j2\pi kn/D} z^{-n} = \frac{1}{D}\sum_{k=0}^{D-1}\sum_{n=-\infty}^{\infty} x[n]\left(e^{j2\pi k/D} z^{-1}\right)^{n}$$

$$= \frac{1}{D}\sum_{k=0}^{D-1} X\left(z e^{-j2\pi k/D}\right)$$

$$X_s\left(e^{j\Omega}\right) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\Omega} e^{-j2\pi k/D}\right) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j(\Omega - 2\pi k/D)}\right)$$



Sampling Rate Conversion

To reduce the number of samples, decimate the sampled signal to form $x_d[n] = x_s[Dn]$. The z transform of $x_d[n]$ is

$$X_d(z) = \sum_{n=-\infty}^{\infty} x_d[n] z^{-n} = \sum_{n=-\infty}^{\infty} x_s[Dn] z^{-n}.$$ Let $m = Dn$.

Then $$X_d(z) = \sum_{\substack{m=-\infty \\ m/D \text{ an} \\ \text{integer}}}^{\infty} x_s[m] z^{-m/D} = \sum_{m=-\infty}^{\infty} x_s[m] z^{-m/D} = X_s\left(z^{1/D}\right)$$

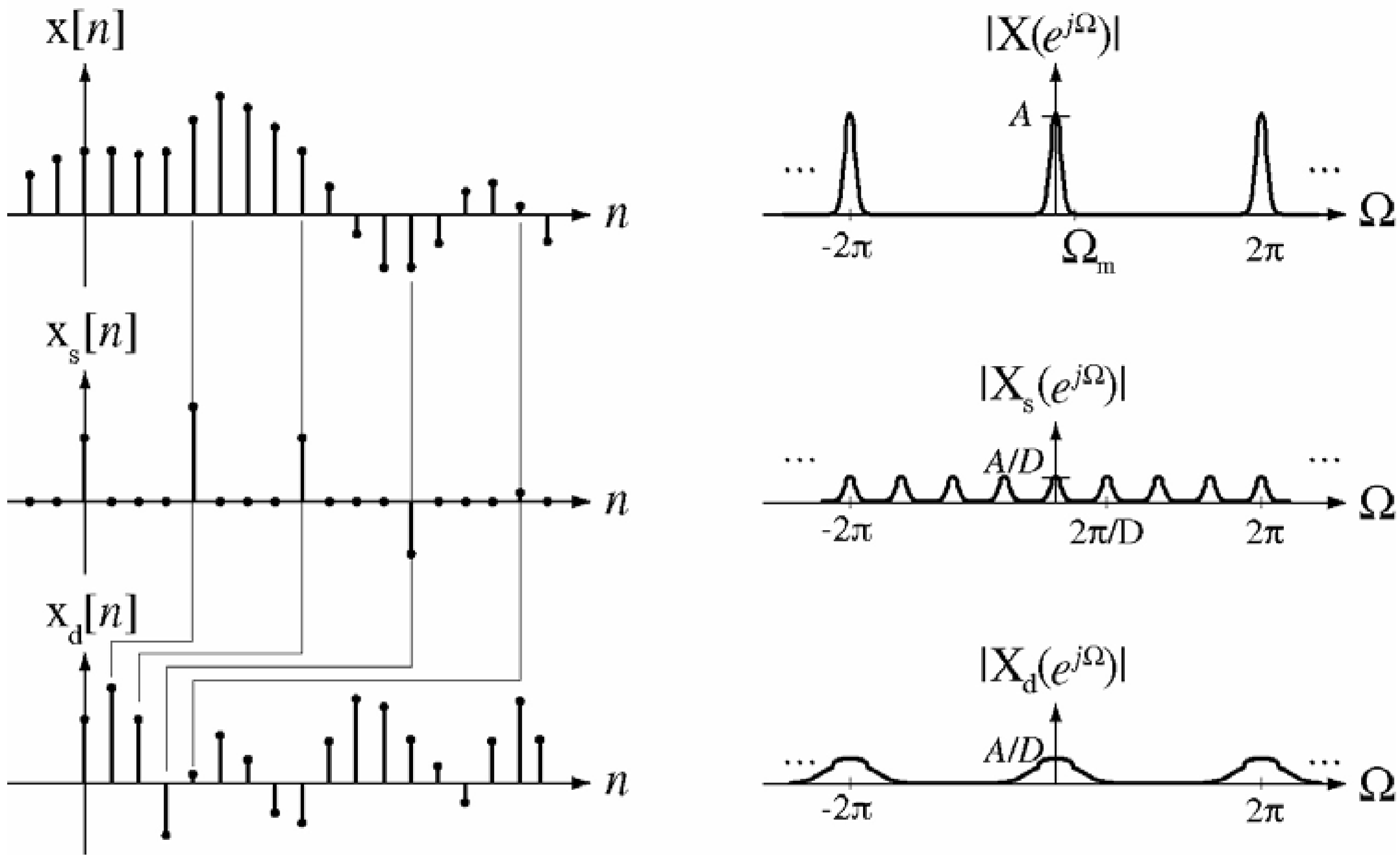
(because all values of $x_s[m]$ for m/D not an integer are zero)

Combining this result with $X_s(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(ze^{-j2\pi k/D}\right)$ we get

$$X_d(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(z^{1/D} e^{-j2\pi k/D}\right)$$

Sampling Rate Conversion

$$X_d\left(e^{j\Omega}\right) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\Omega/D}e^{-j2\pi k/D}\right) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j\left(\Omega-2\pi k\right)/D}\right)$$



Sampling Rate Conversion

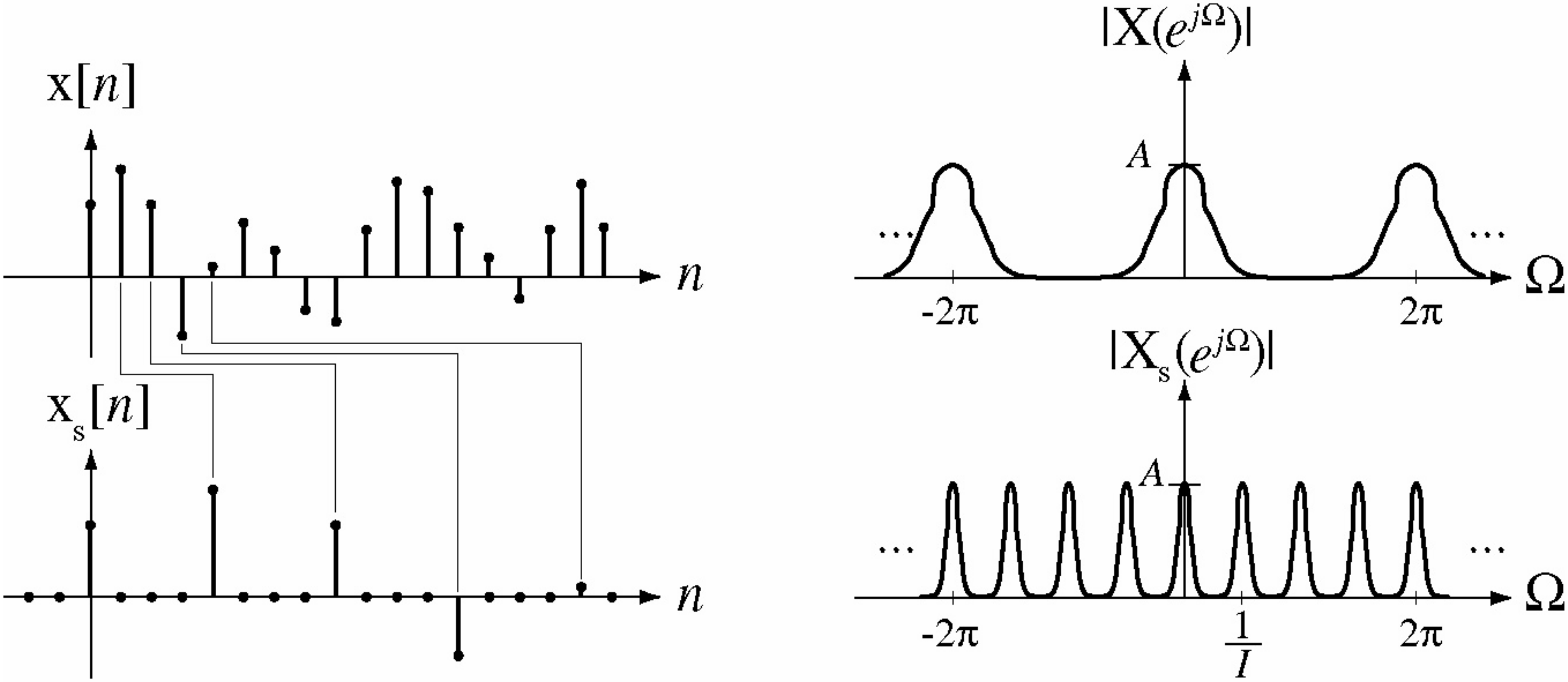
The opposite of downsampling is upsampling. If the original signal is $\mathrm{x}[n]$ the upsampled signal is

$$\mathrm{x}_s[n] = \begin{cases} \mathrm{x}[n/I] & , \ n/I \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

Sampling Rate Conversion

A discrete-time expansion by a factor of I corresponds to a discrete-time-frequency compression by the same factor.

$$X_s(z) = X(z^I) \Rightarrow X_s(e^{j\Omega}) = X(e^{jI\Omega})$$



Sampling Rate Conversion

An ideal lowpass filter with transfer function

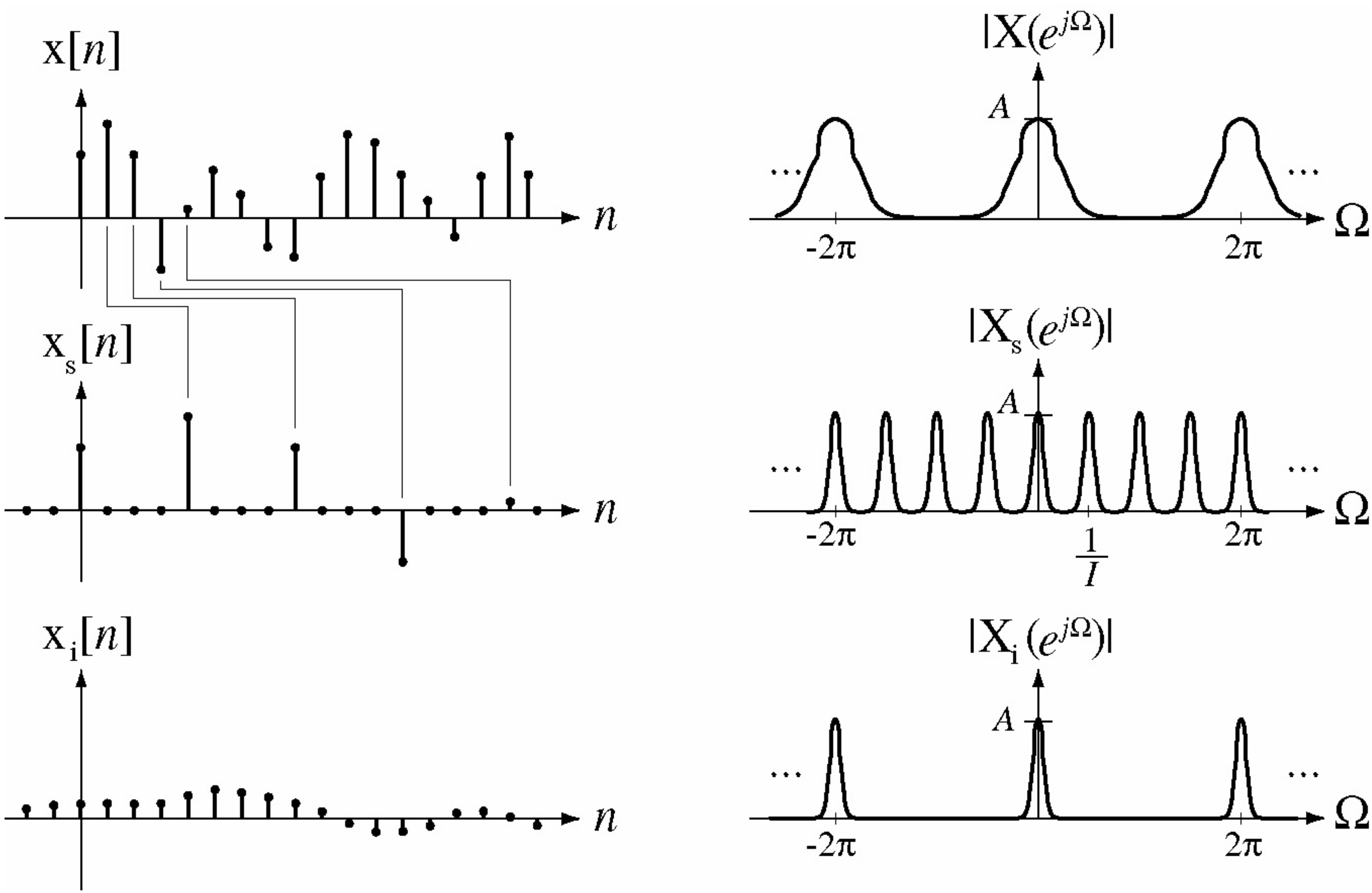
$$\mathrm{H}\left(e^{j\Omega}\right)=\sum_{k=-\infty}^{\infty}\mathrm{rect}\left(I\left(\Omega-2\pi k\right)/2\pi\right)=\begin{cases}1\ ,\ \ \left|\Omega\right|<\pi/I\\0\ ,\ \ \pi/I<\left|\Omega\right|<\pi\end{cases}$$

could be used to interpolate between sample values yielding

$$\mathrm{X}_i\left(e^{j\Omega}\right)=\mathrm{X}_s\left(e^{j\Omega}\right)\sum_{k=-\infty}^{\infty}\mathrm{rect}\left(I\left(\Omega-2\pi k\right)/2\pi\right)$$

which corresponds to $\mathrm{x}_i\left[n\right]=\mathrm{x}_s\left[n\right]*\left(1/I\right)\mathrm{sinc}\left(n/I\right)$ in the time domain.

Sampling Rate Conversion



Polyphase Filters

The polyphase filter was developed for the efficient implementation of sampling rate conversion. Any transfer function is of the form

$$\mathrm{H}(z) = \cdots + \mathrm{h}[0] + z^{-1}\,\mathrm{h}[1] + \cdots + z^{-k}\,\mathrm{h}[k] + \cdots$$

which can be regrouped and written as

$$\begin{aligned}
\mathrm{H}(z) = \cdots + \mathrm{h}[0] \qquad & + z^{-M}\,\mathrm{h}[M] + \cdots \\
\cdots z^{-1}\mathrm{h}[1] \qquad & + z^{-(M+1)}\,\mathrm{h}[M+1] + \cdots \\
\vdots \qquad & \qquad \vdots \\
\cdots z^{-(M-1)}\,\mathrm{h}[M-1] & + z^{-(2M-1)}\,\mathrm{h}[2M-1] + \cdots \\
\vdots \qquad & \qquad \vdots
\end{aligned}$$

Polyphase Filters

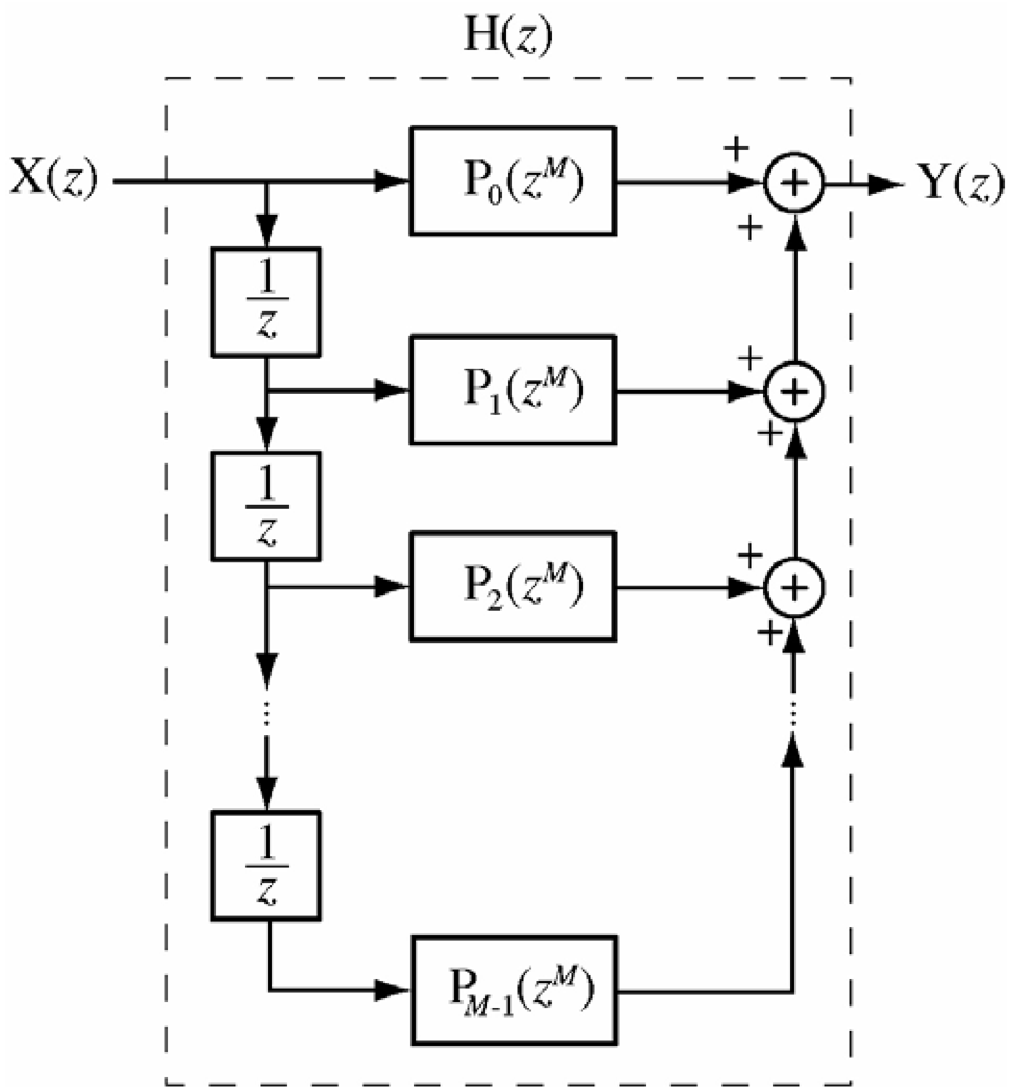
$$
\begin{aligned}
\mathrm{H}(z) = \cdots + \mathrm{h}[0] & \qquad + z^{-M}\,\mathrm{h}[M] + \cdots \\
\cdots z^{-1}\big(\mathrm{h}[1] & \qquad + z^{-M}\,\mathrm{h}[M+1]\big) + \cdots \\
\vdots & \qquad \vdots \\
\cdots z^{-(M-1)}\big(\mathrm{h}[M-1] & + z^{-M}\,\mathrm{h}[2M-1]\big) + \cdots \\
\vdots & \qquad \vdots
\end{aligned}
$$

This can be written compactly as

$$\mathrm{H}(z) = \sum_{i=0}^{M-1} z^{-1}\,\mathrm{P}_i\left(z^M\right) \text{ where } \mathrm{P}_i(z) = \sum_{n=-\infty}^{\infty} \mathrm{h}(nM+i)\,z^{-n}$$

This is called the “M-component polyphase decomposition” of $\mathrm{H}(z)$ and the $\mathrm{P}_i(z)$'s are the polyphase components of $\mathrm{H}(z)$.

Polyphase Filters



The Noble Identities

The relationships between a signal and a decimated version of the signal was found to be $X\left(e^{j\Omega}\right) = D\,X_d\left(e^{jD\Omega}\right)$, $0 < \left|\Omega\right| < \pi / D$.

It then follows that, in the z domain,

$$X\left(z\right) = D\,X_d\left(z^D\right) \text{ and } X_d\left(z\right) = \left(1/D\right)X\left(z^{1/D}\right)$$

for signals sampled according to the sampling theorem.

Consider a downsampler followed by a filter with input signal $x\left[n\right]$ and output signal $y\left[n\right]$ and let the downsampler output be $y_1\left[n\right]$.

The Noble Identities

$Y_1(z) = (1/D)X(z^{1/D})$ and $Y(z) = H(z)Y_1(z)$

Therefore $Y(z) = (1/D)H(z)X(z^{1/D})$.

Now reverse the order of the downsampler and filter.

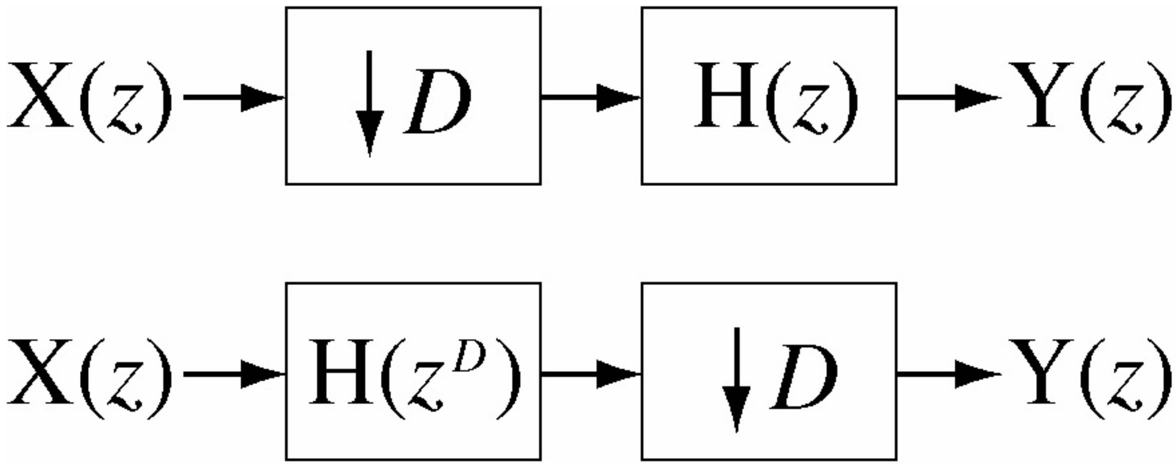
$Y_1(z) = H(z)X(z)$ and $Y(z) = (1/D)Y_1(z^{1/D})$

Therefore $Y(z) = (1/D)H(z^{1/D})X(z^{1/D})$.

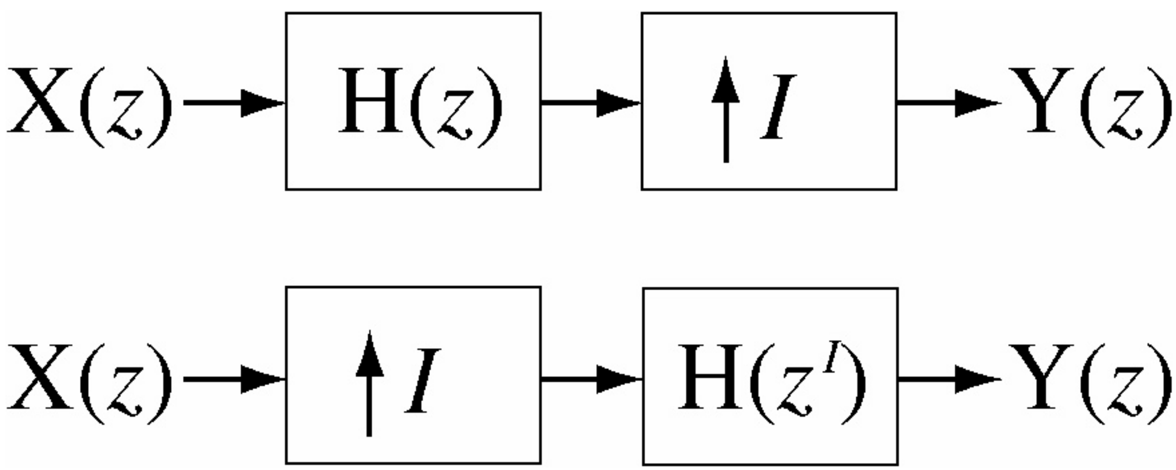
The two output signals are not the same because the downsampler is not an LTI system. But they would be the same if $z \to z^D$ in $H(z)$ in the second system.

The Noble Identities

For decimators



For interpolators



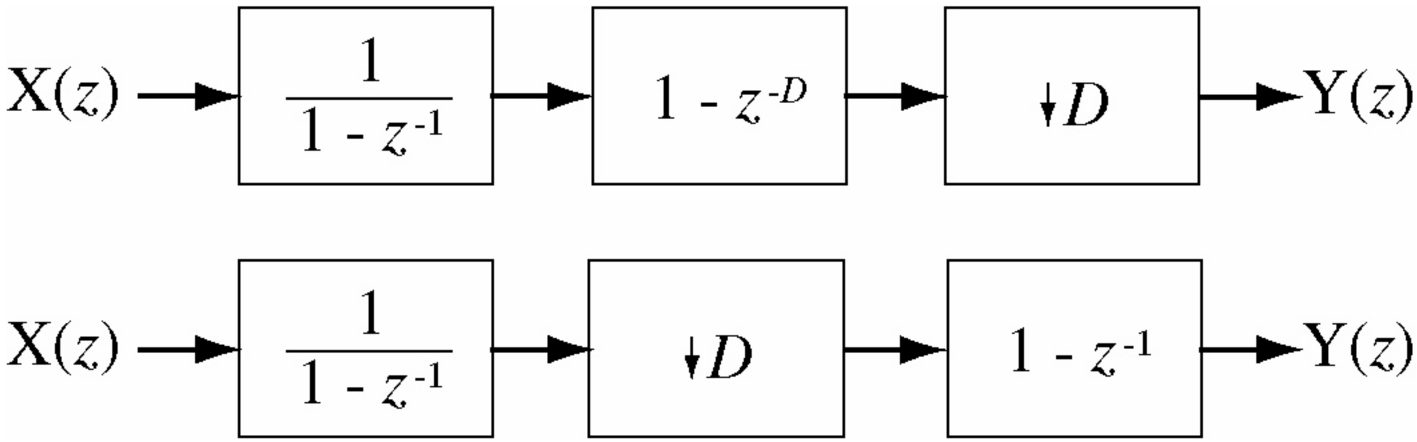
Cascaded Integrator Comb Filters

In both decimation and interpolation a lowpass filter is needed. The cascaded integrator comb filter is an efficient structure for lowpass filtering in decimation or interpolation. Its transfer function is

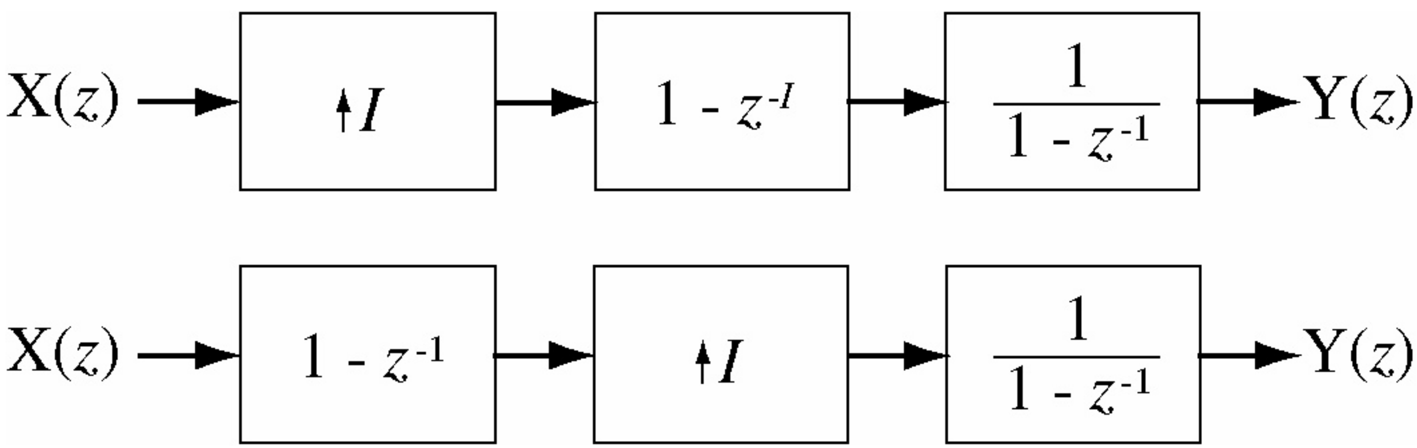
$$\mathrm{H}(z) = \sum_{k=0}^{M-1} z^{-k} = \frac{1 - z^{-M}}{1 - z^{-1}}$$

Cascaded Integrator Comb Filters

In decimation



In interpolation



Cascaded Integrator Comb Filters

The transfer function of the cascaded integrator filter can be written as

$$H(z)=\sum_{k=0}^{M-1} z^{-k}=1+z^{-1}+z^{-2}+\cdots+z^{-(M-1)}$$

For $M=4,\ H(z)=\sum_{k=0}^{M-1} z^{-k}=1+z^{-1}+z^{-2}+z^{-3}=\left(1+z^{-1}\right)\left(1+z^{-2}\right).$

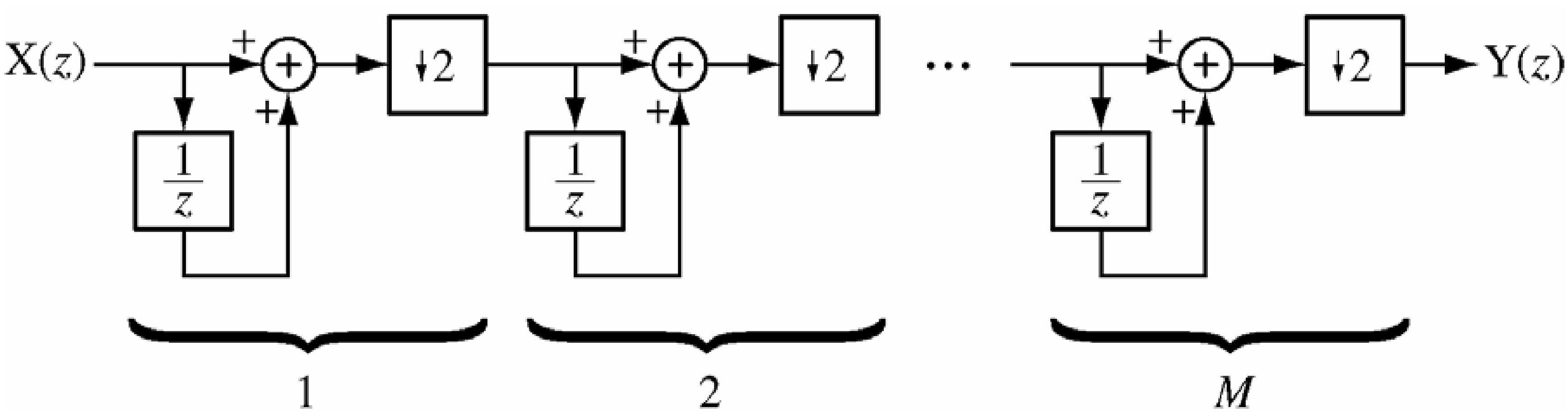
For $M=8,\ H(z)=1+z^{-1}+z^{-2}+z^{-3}+z^{-4}+z^{-5}+z^{-6}+z^{-7}$

$$=\left(1+z^{-1}\right)\left(1+z^{-2}\right)\left(1+z^{-4}\right)$$

For $M=2^{k},\ H(z)=\left(1+z^{-1}\right)\left(1+z^{-2}\right)\cdots\left(1+z^{-2^{(k-1)}}\right)$

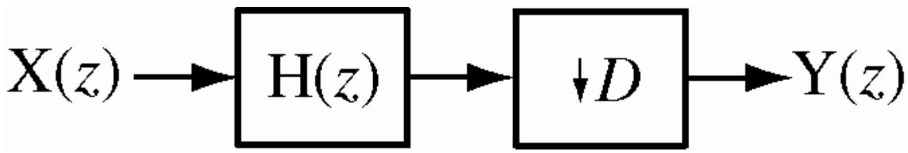
Cascaded Integrator Comb Filters

$$H(z) = \left(1 + z^{-1}\right)\left(1 + z^{-2}\right)\cdots\left(1 + z^{-2^{(k-1)}}\right)$$



Polyphase Structures in Decimation and Interpolation

The basic decimator model is a lowpass filter followed by a downsampler.

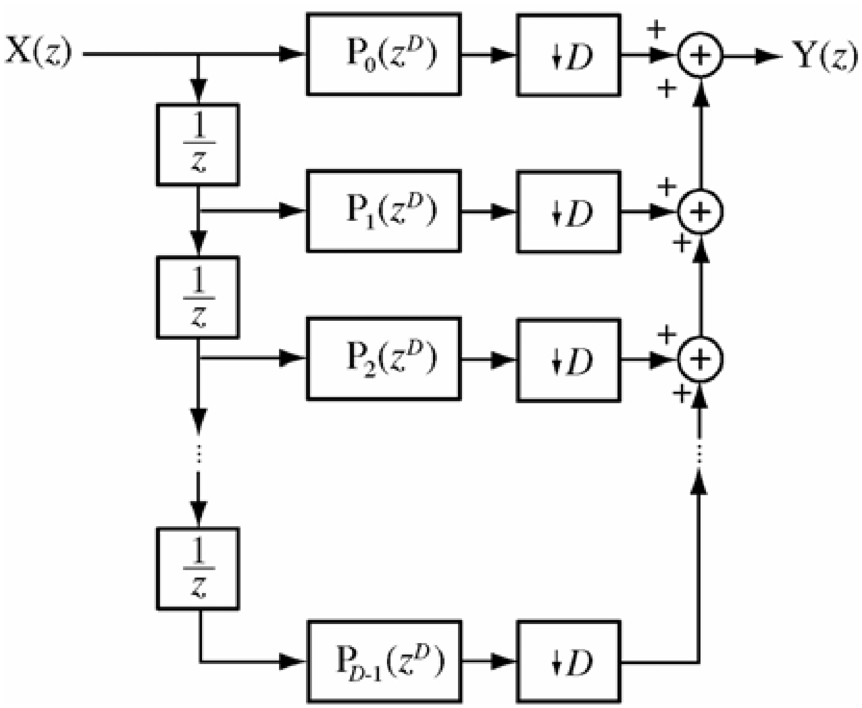


One inefficiency is that the filtering computations are done at the higher sampling rate but the results are only needed at the lower sampling rate. We can improve the efficiency by using a polyphase structure.

Polyphase Structures in Decimation and Interpolation

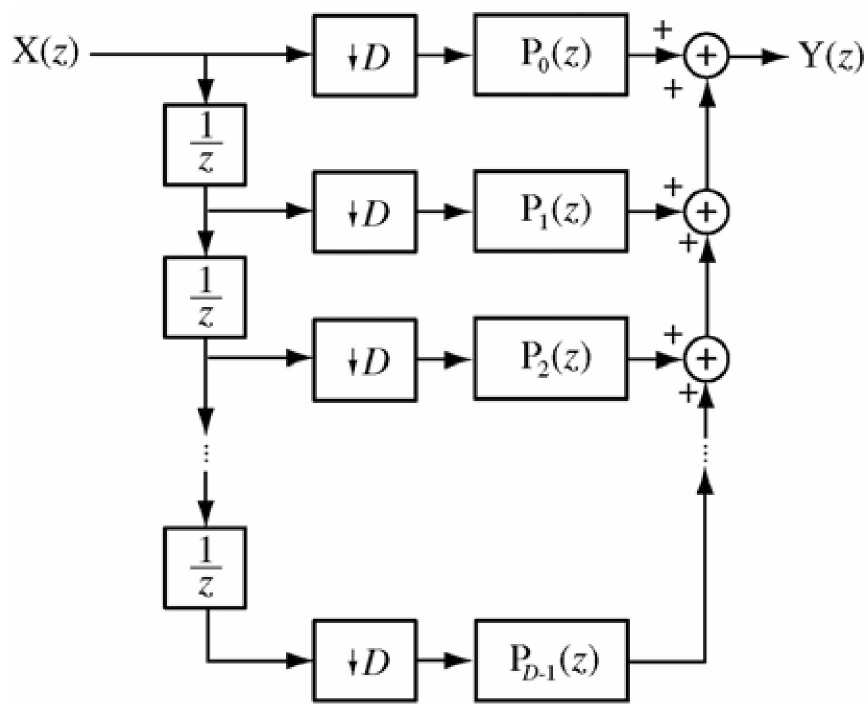
Filter → Decimate

Less Efficient



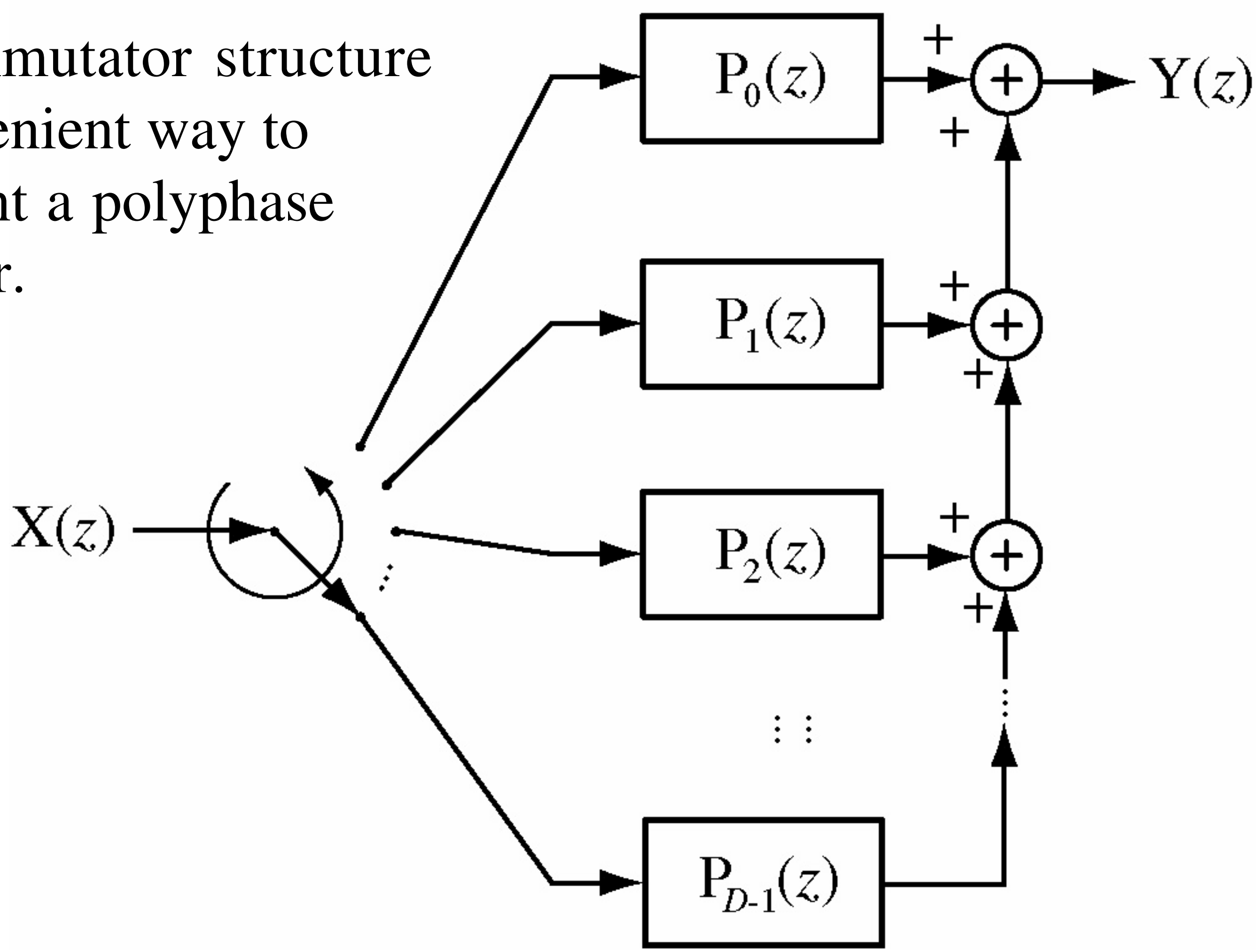
Decimate → Filter

More Efficient



Polyphase Structures in Decimation and Interpolation

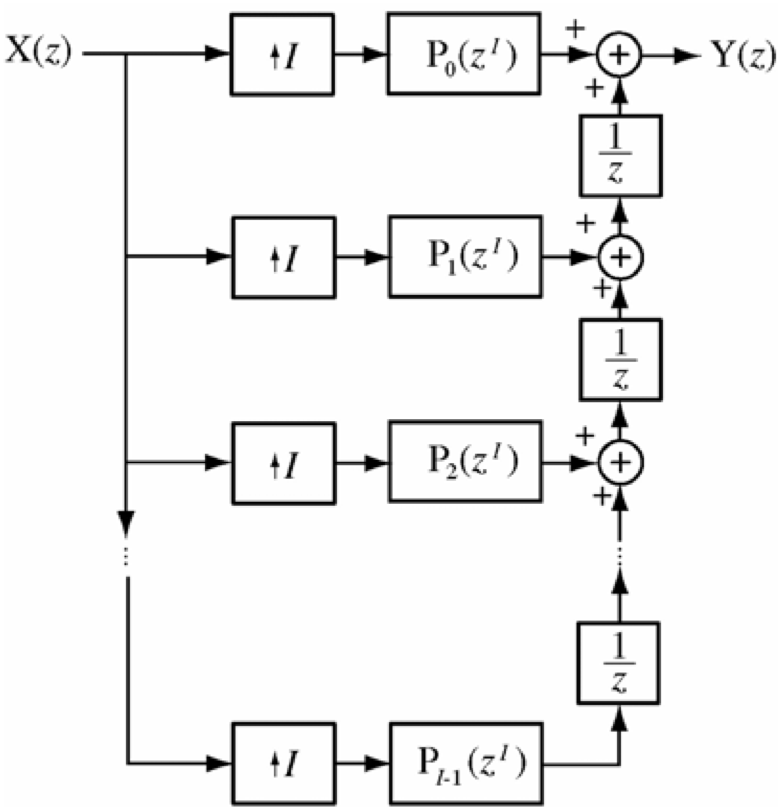
This commutator structure is a convenient way to implement a polyphase decimator.



Polyphase Structures in Decimation and Interpolation

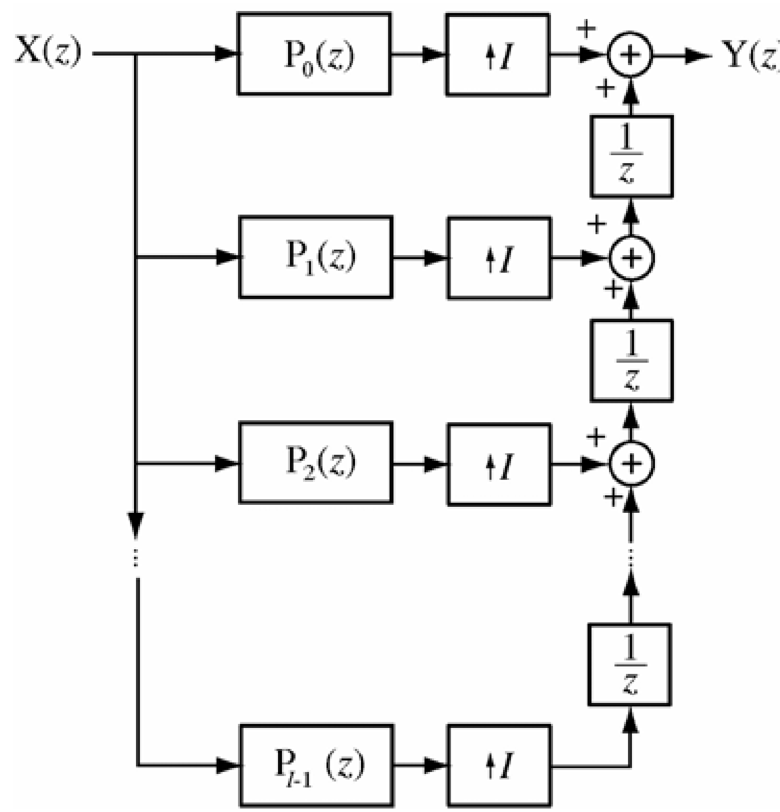
Interpolate → Filter

Less Efficient

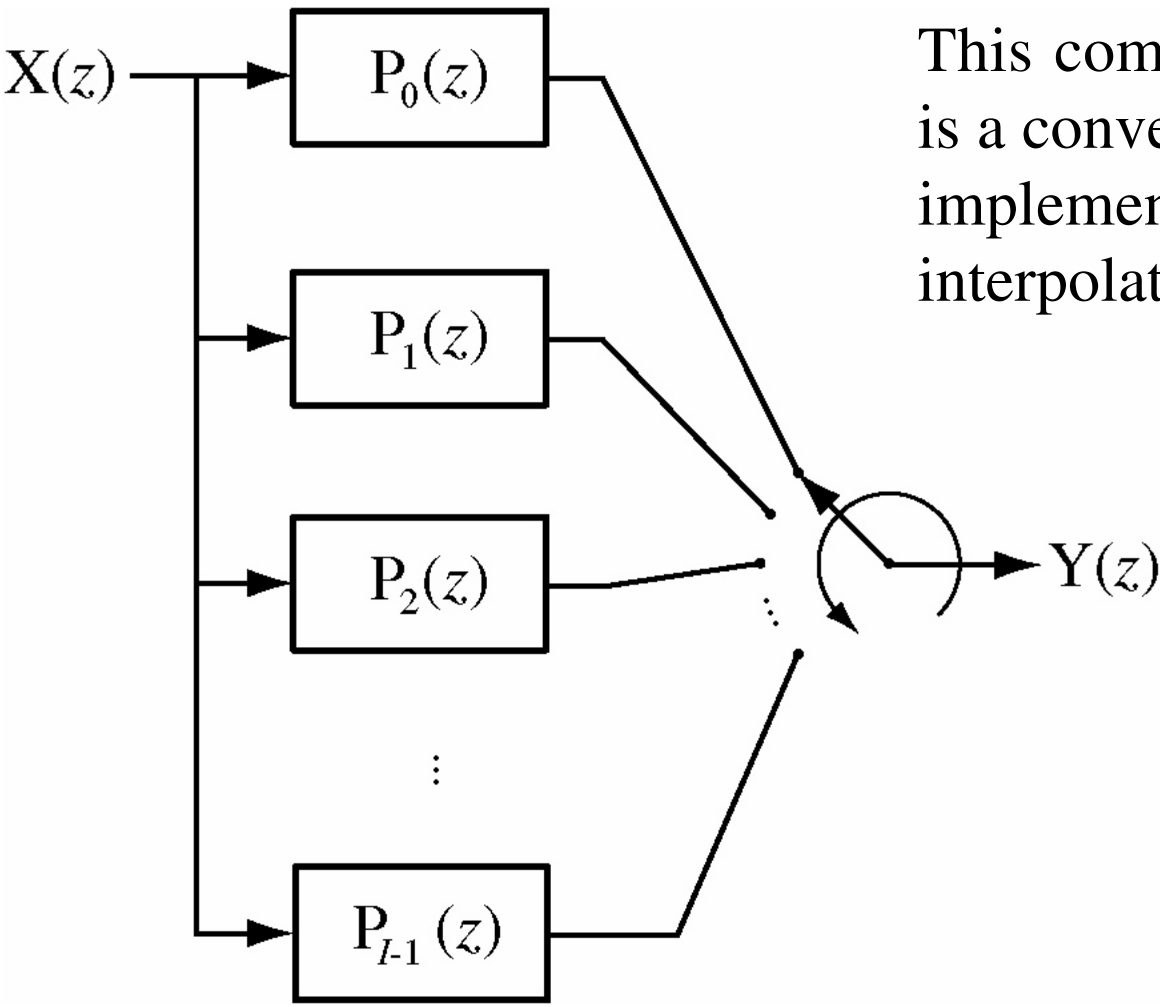


Filter → Interpolate

More Efficient



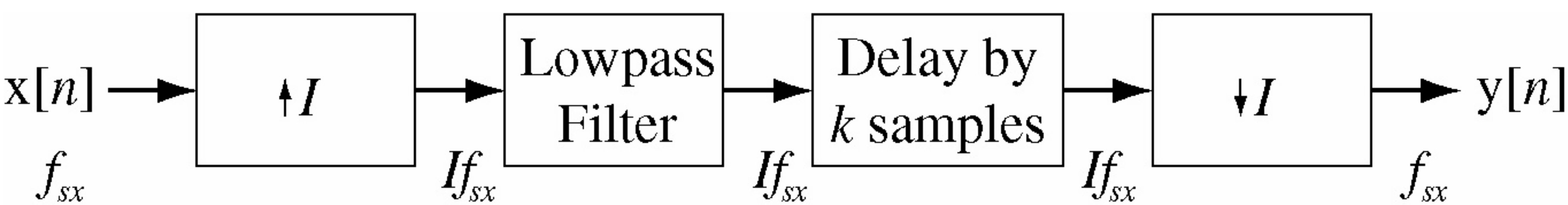
Polyphase Structures in Decimation and Interpolation



This commutator structure is a convenient way to implement a polyphase interpolator.

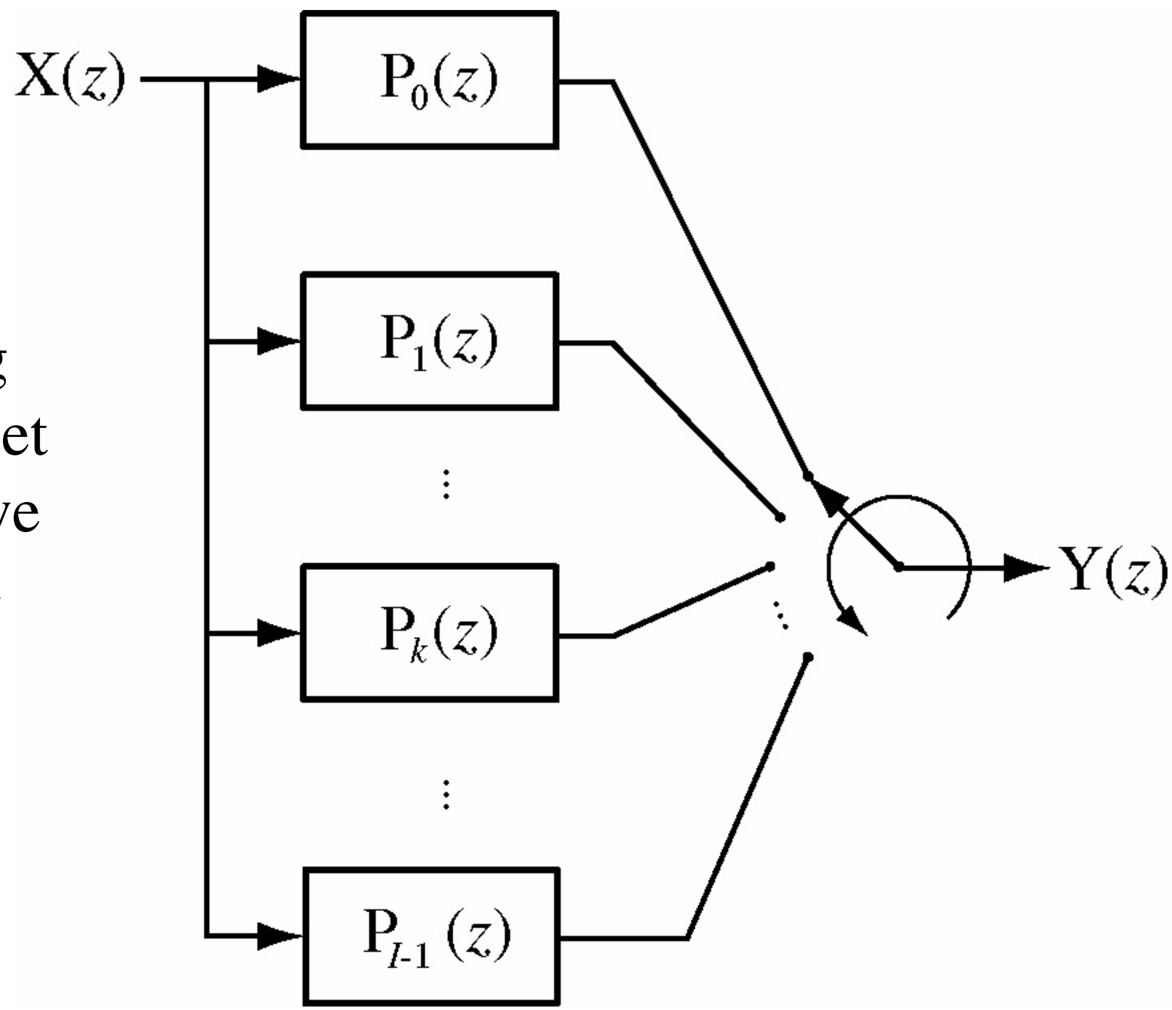
Phase Shifter

Sometimes it is desired to delay a signal $\mathrm{x}[n]$ by a fraction of the time between samples T_{sx}. If that fraction is k / I and k and I are both integers, the delay can be acccomplished by sample-rate conversion methods.



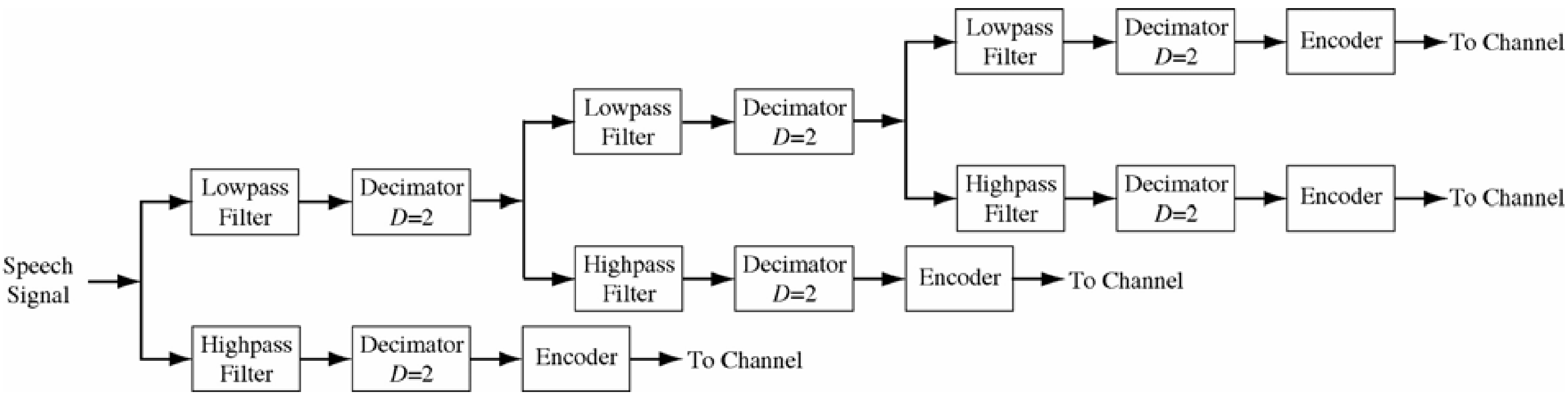
Phase Shifter

The delay can also be accomplished efficiently by using an interpolating polyphase commutator set at the *k*th delay. Since we are only interested in the *k*th delay we only need the *k*th polyphase filter.



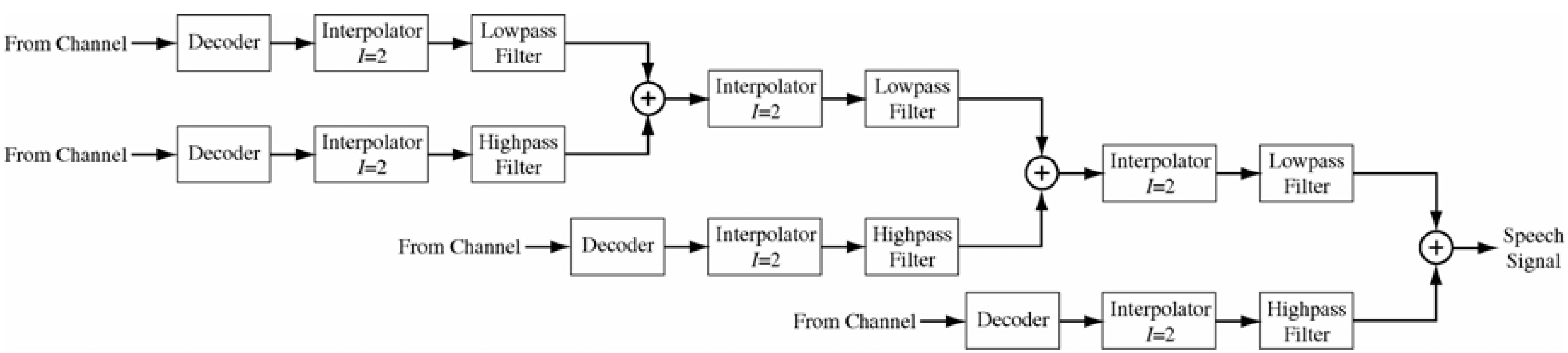
Subband Coding

Because most of the power of speech signals is at the lower frequencies it is efficient to use more bits on the low frequency part of the spectrum. Subband coding accomplishes that by filtering and decimating multiple times.



Subband Coding

The speech signal can be recovered by doing the opposite of subband coding.



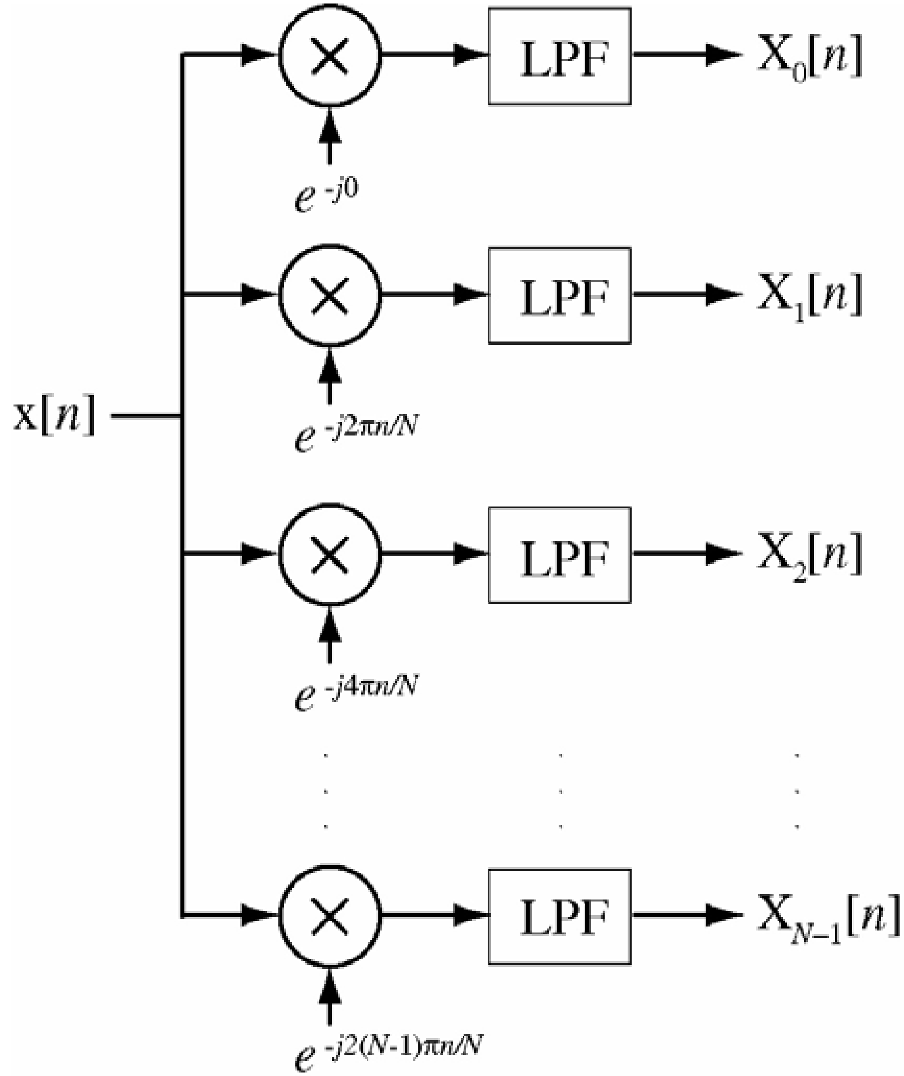
Digital Filter Banks

An important type of analysis filter bank is the DFT filter bank. Let the lowpass filters have impulse responses

$$\mathrm{h}[n] = \sum_{m=0}^{N-1} \delta[n-m] \text{ and}$$

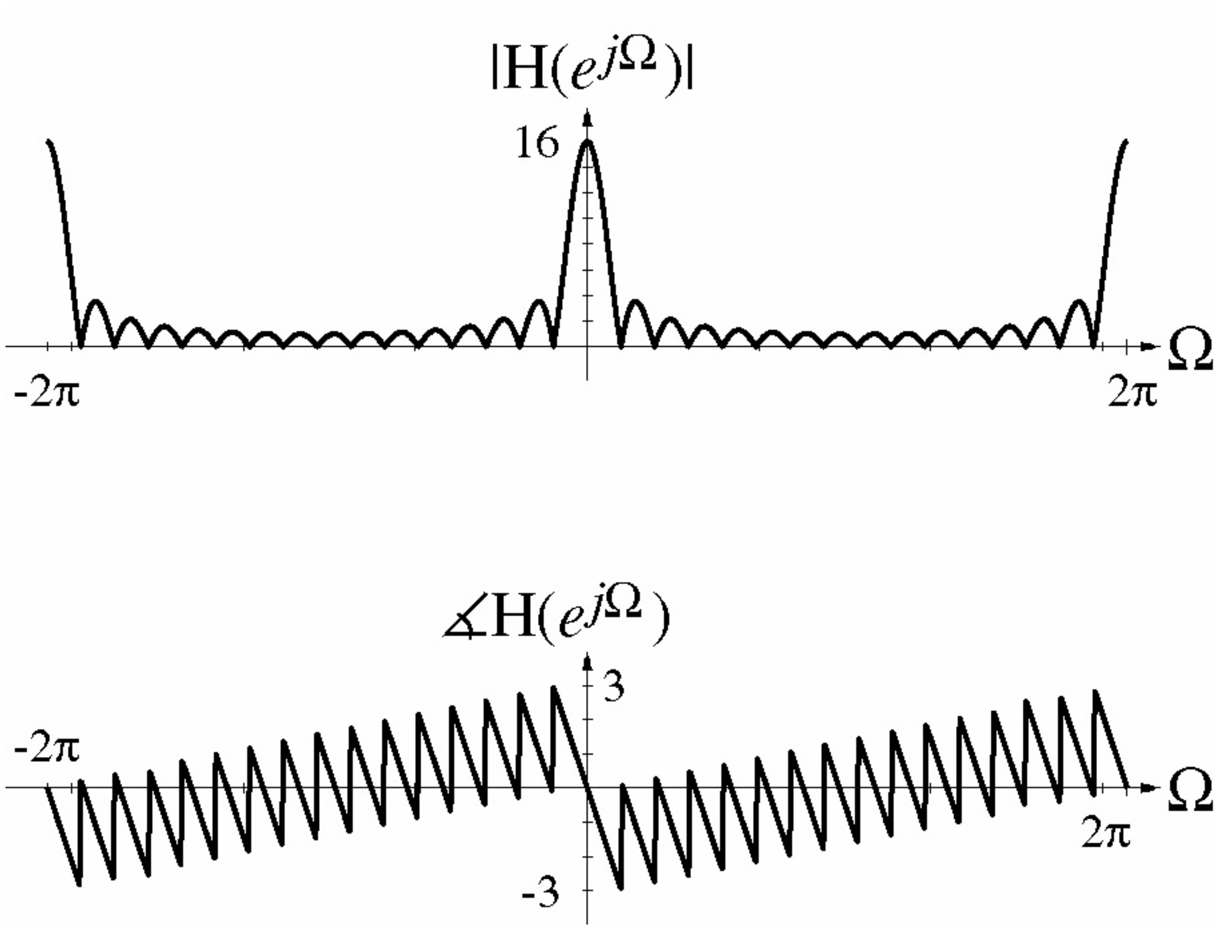
frequency response

$$\mathrm{H}\left(e^{j\Omega}\right) = e^{-j\Omega(N-1)/2} \frac{\sin(N\Omega/2)}{\sin(\Omega/2)}$$



Digital Filter Banks

$$\mathrm{H}\left(e^{j\Omega}\right) = e^{-j\Omega(N-1)/2}\,\frac{\sin\left(N\Omega/2\right)}{\sin\left(\Omega/2\right)}$$



Digital Filter Banks

The output signal from the kth filter would be

$$Y_k\left(e^{j\Omega}\right)=\left(X\left(e^{j\Omega}\right)*\delta\left(\Omega-2\pi k/N\right)\right)e^{-j\Omega(N-1)/2}\frac{\sin\left(N\Omega/2\right)}{\sin\left(\Omega/2\right)}$$

$$Y_k\left(e^{j\Omega}\right)=X\left(e^{j(\Omega-2\pi k/N)}\right)e^{-j\Omega(N-1)/2}\frac{\sin\left(N\Omega/2\right)}{\sin\left(\Omega/2\right)}$$

or, in the time domain

$$y_k[n]=\left(x[n]e^{-j2\pi kn/N}\right)*h[n]=\sum_{m=n-(N-1)}^{n}x[n-m]e^{-j2\pi k(n-m)/N}$$

$y_k[N-1]=\sum_{q=0}^{N-1}x[q]e^{-j2\pi kq/N}$ which, at any time n, is $X[k]$ the kth harmonic value in the DFT of the last N values of n.

Digital Filter Banks

Since the filters are lowpass, the output signals can be decimated by N. Without decimation they are

$$\mathrm{y}_k[n] = \left(\mathrm{x}[n]e^{-j2\pi kn/N}\right) * \mathrm{h}_0[n] = \sum_{m=n-(N-1)}^{n} \mathrm{h}_0[n-m]\,\mathrm{x}[m]\,e^{-j2\pi km/N}$$

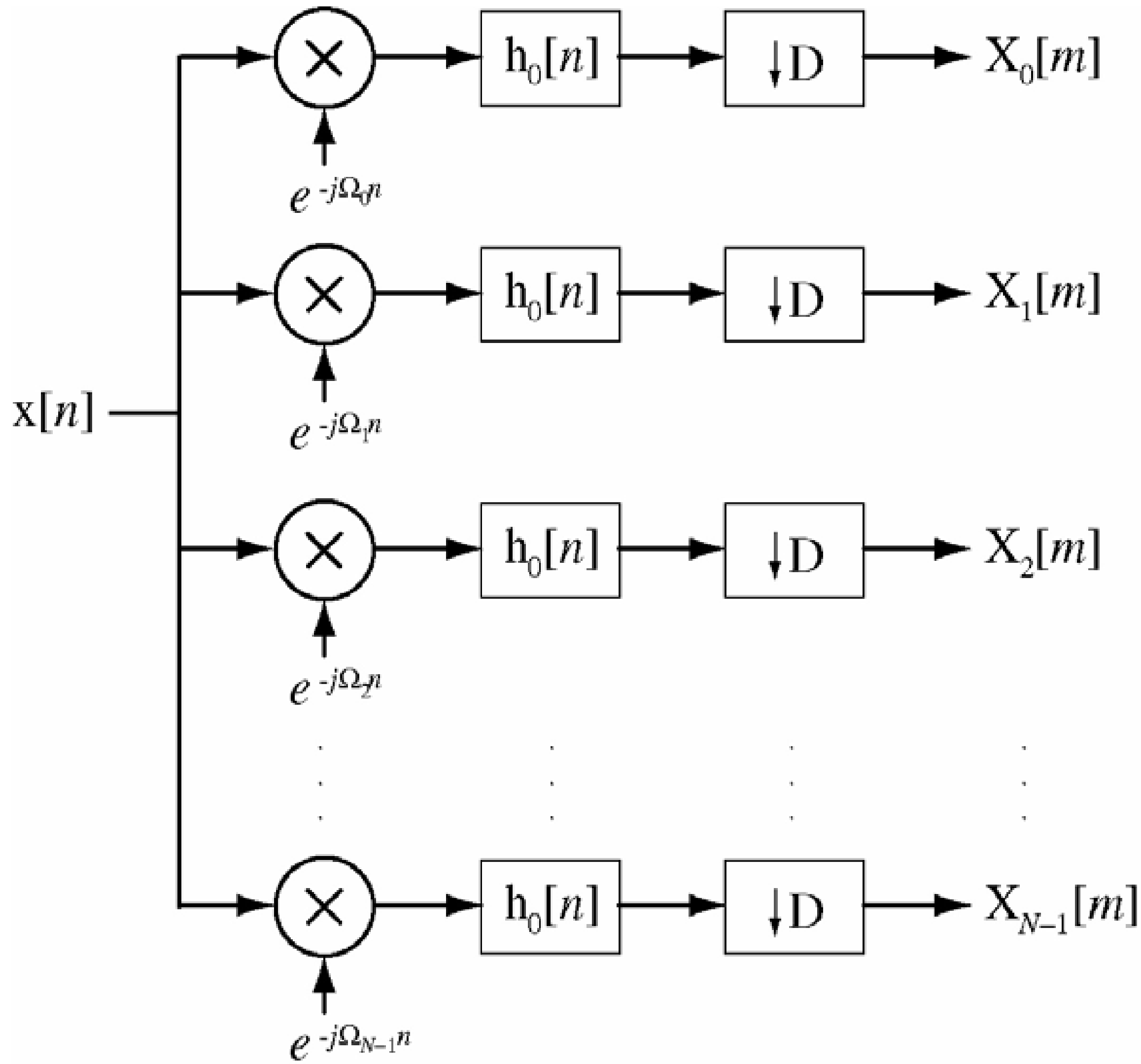
and with decimation they are

$$\mathrm{X}_k[m] = \sum_{n=m-(N-1)}^{m} \mathrm{h}_0[mN-n]\,\mathrm{x}[n]\,e^{-j2\pi kn/N}$$

where m is the discrete time at the output which is not the same as n the discrete time at the input.

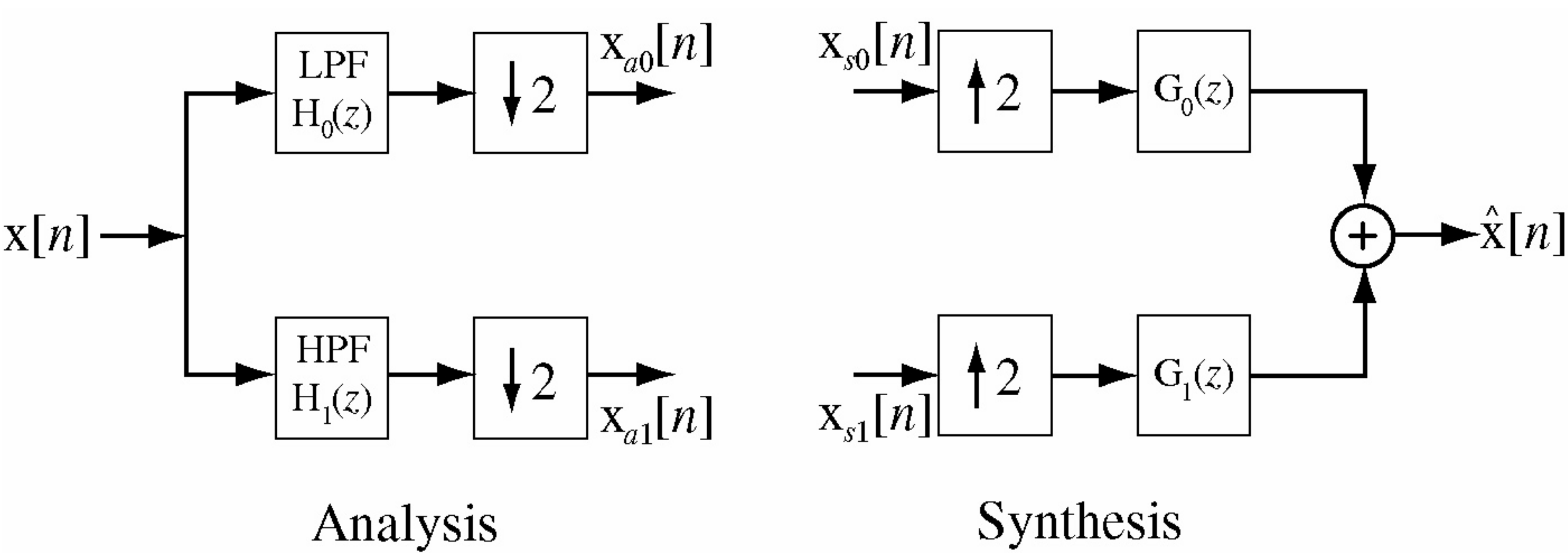
Digital Filter Banks

An analysis filter bank with decimation.



Quadrature Mirror Filters

The two-channel quadrature mirror structure



Quadrature Mirror Filters

The Fourier transforms of the output signals from the analysis section are

$$X_{a0}\left(e^{j\Omega}\right)=\frac{1}{2}\left[X\left(e^{j\Omega/2}\right)H_0\left(e^{j\Omega/2}\right)+X\left(e^{j(\Omega-2\pi)/2}\right)H_0\left(e^{j(\Omega-2\pi)/2}\right)\right]$$

and

$$X_{a1}\left(e^{j\Omega}\right)=\frac{1}{2}\left[X\left(e^{j\Omega/2}\right)H_1\left(e^{j\Omega/2}\right)+X\left(e^{j(\Omega-2\pi)/2}\right)H_1\left(e^{j(\Omega-2\pi)/2}\right)\right]$$

The Fourier transform of the output signal from the synthesis section is

$$\hat{X}\left(e^{j\Omega}\right)=X_{s0}\left(e^{j2\Omega}\right)G_0\left(e^{j\Omega}\right)+X_{s1}\left(e^{j2\Omega}\right)G_1\left(e^{j\Omega}\right)$$

Quadrature Mirror Filters

If we connect the outputs from the analysis section to the inputs of the synthesis section we get

$$\hat{X}\left(e^{j\Omega}\right) = \frac{1}{2}\left[X\left(e^{j\Omega}\right)H_0\left(e^{j\Omega}\right) + X\left(e^{j(\Omega-\pi)}\right)H_0\left(e^{j(\Omega-\pi)}\right)\right]G_0\left(e^{j\Omega}\right)$$
$$+\frac{1}{2}\left[X\left(e^{j\Omega}\right)H_1\left(e^{j\Omega}\right) + X\left(e^{j(\Omega-\pi)}\right)H_1\left(e^{j(\Omega-\pi)}\right)\right]G_1\left(e^{j\Omega}\right)$$

which can be written as $\hat{X}\left(e^{j\Omega}\right) = Q\left(e^{j\Omega}\right)X\left(e^{j\Omega}\right) + A\left(e^{j\Omega}\right)X\left(e^{j(\Omega-\pi)}\right)$

or in the z domain as $\hat{X}(z) = Q(z)X(z) + A(z)X(-z)$

where

$$Q(z) = (1/2)\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]$$
$$A(z) = (1/2)\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]$$

Quadrature Mirror Filters

The second term $A(z)X(-z)$ is an undesirable alias. If we set $A(z)=0$ we get

$$H_0(-z)G_0(z)+H_1(-z)G_1(z)=0$$

$$H_0\left(e^{j(\Omega-\pi)}\right)G_0\left(e^{j\Omega}\right)=-H_1\left(e^{j(\Omega-\pi)}\right)G_1\left(e^{j\Omega}\right)$$

Then if we set

$$G_0\left(e^{j\Omega}\right)=H_1\left(e^{j(\Omega-\pi)}\right) \quad \text{and} \quad G_1\left(e^{j\Omega}\right)=-H_0\left(e^{j(\Omega-\pi)}\right)$$

we get

$$H_0\left(e^{j(\Omega-\pi)}\right)H_1\left(e^{j(\Omega-\pi)}\right)=H_1\left(e^{j(\Omega-\pi)}\right)H_0\left(e^{j(\Omega-\pi)}\right)$$

and $A(z)=0$.

Quadrature Mirror Filters

If $\mathrm{H}\left(e^{j\Omega}\right)$ is lowpass

and we set

$\mathrm{H}_0\left(e^{j\Omega}\right) = \mathrm{H}\left(e^{j\Omega}\right) = \mathrm{H}_1\left(e^{j(\Omega-\pi)}\right)$

then $\mathrm{H}_1\left(e^{j\Omega}\right)$ is highpass

and is a "mirror image"

of $\mathrm{H}_0\left(e^{j\Omega}\right)$. The corresponding

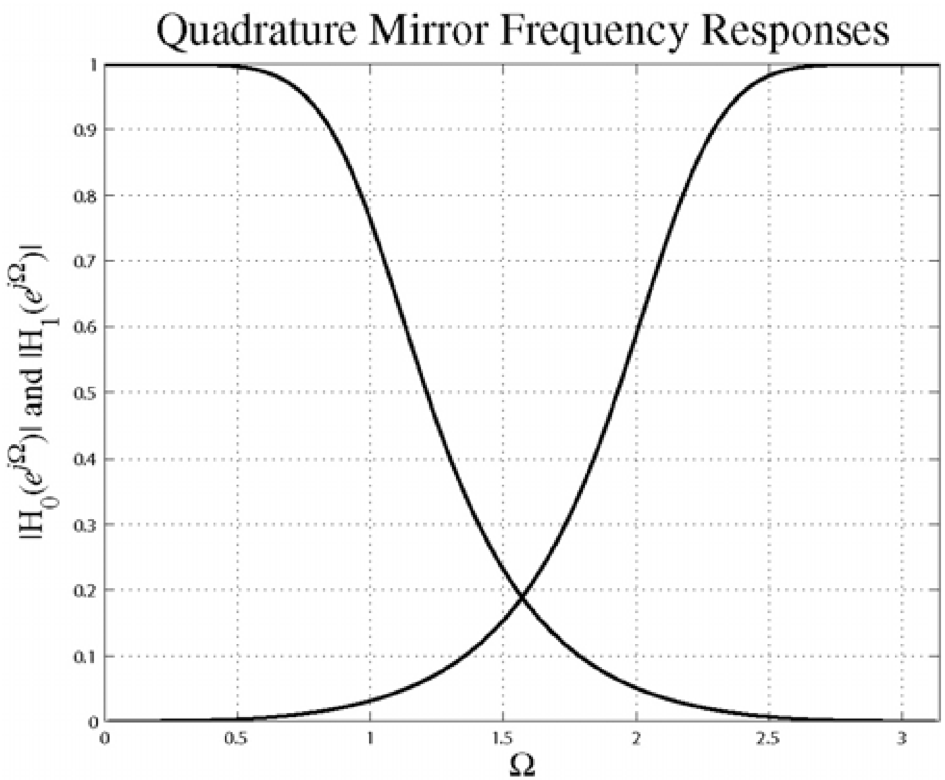
impulse responses are

$\mathrm{h}_0[n] = \mathrm{h}[n]$ and

$\mathrm{h}_1[n] = (-1)^n \mathrm{h}[n]$. Then if

$\mathrm{G}_0(z) = \mathrm{H}(z)$, $\mathrm{G}_1(z) = -\mathrm{H}(-z)$

for elimination of aliases.



Quadrature Mirror Filters

In summary, for elimination of aliases

$$H_0(z) = H(z)$$

$$H_1(z) = H(-z)$$

$$G_0(z) = H(z)$$

$$G_1(z) = -H(-z)$$

Quadrature Mirror Filters

One frequent use of quadrature mirror filters is to break a signal into multiple parts with the analysis section, analyze the parts and then reconstruct the signal in the synthesis section. What is the requirement for perfect reconstruction? Since we know that

$$\hat{X}(z) = Q(z)X(z) + A(z)X(-z)$$

and we want the synthesized signal to be a delayed version of the original signal we want $\hat{X}(z) = z^{-k}X(z)$. If we have already eliminated the aliases we have

$$z^{-k}X(z) = Q(z)X(z)$$

$$Q(z) = (1/2)\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right] = z^{-k}$$

Quadrature Mirror Filters

Combining

$$z^{-k}\,X(z) = Q(z)X(z)$$

$$Q(z) = (1/2)\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right] = z^{-k}$$

with the elimination of aliases we get

$$(1/2)\left[H(z)H(z) - H(-z)H(-z)\right] = z^{-k}$$

or

$$H^2(z) - H^2(-z) = 2z^{-k} \Rightarrow H^2\left(e^{j\Omega}\right) - H^2\left(e^{j(\Omega-\pi)}\right) = 2e^{-jk\Omega}$$

which implies that

$$\left|H^2\left(e^{j\Omega}\right) - H^2\left(e^{j(\Omega-\pi)}\right)\right| = 2$$

for perfect reconstruction.

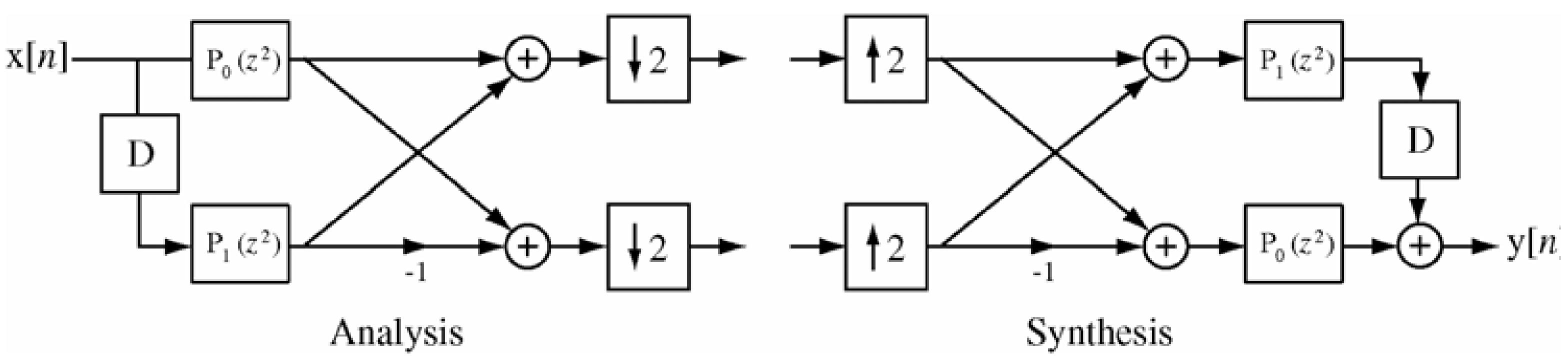
Polyphase Quadrature Mirror Filters

Let

$$H_0(z) = P_0(z^2) + z^{-1}P_1(z^2)$$
$$H_1(z) = P_0(z^2) - z^{-1}P_1(z^2)$$

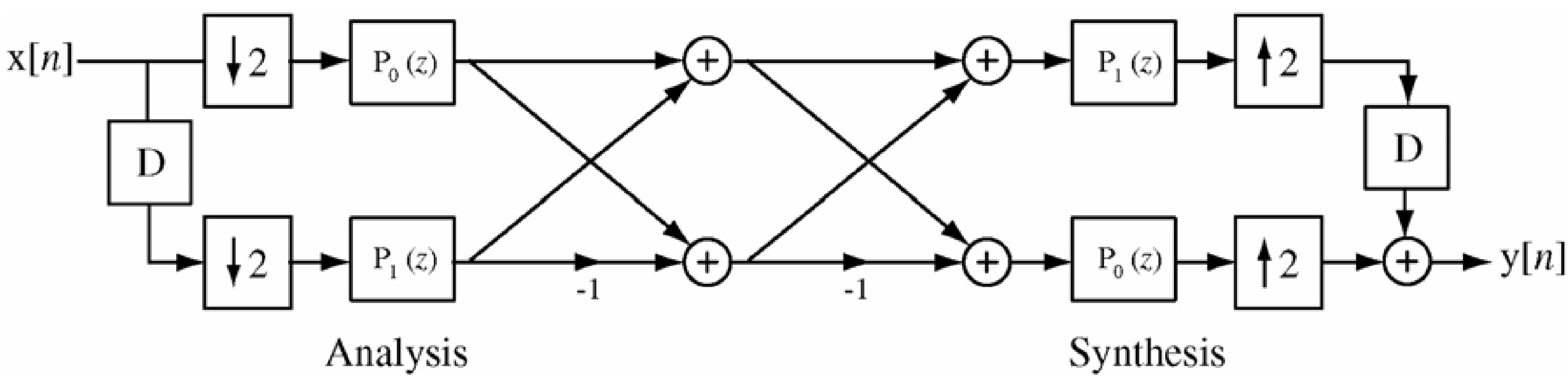
Then

$$G_0(z) = P_0(z^2) + z^{-1}P_1(z^2)$$
$$G_1(z) = -\left[P_0(z^2) - z^{-1}P_1(z^2)\right]$$



Polyphase Quadrature Mirror Filters

If the analysis and synthesis sections are cascade connected and the noble identities are used to make the computations more efficient we get



Perfect Reconstruction QMF

Perfect reconstruction of the input signal can be acheived by an FIR half-band filter of length $2N-1$. A half-band filter is a zero-phase FIR filter whose impulse response $\mathrm{b}[n]$ satisfies

$$\mathrm{b}[2n] = \begin{cases} \text{constant} & , \ n = 0 \\ 0 & , \ n \neq 0 \end{cases}$$

If it is zero-phase then $\mathrm{b}[n] = \mathrm{b}[-n]$. The frequency response is $\mathrm{B}\left(e^{j\Omega}\right) = \sum_{n=-K}^{K} \mathrm{b}[n] e^{-j\Omega n}$ where K is odd.

Perfect Reconstruction QMF

$$\mathrm{B}\left(e^{j\Omega}\right) = \sum_{n=-K}^{K} \mathrm{b}[n] e^{-j\Omega n} = \mathrm{b}[0] + 2\sum_{\substack{n=1 \\ n \text{ odd}}}^{K} \mathrm{b}[n]\cos(\Omega n)$$

$$\mathrm{B}\left(e^{j\Omega}\right) = \mathrm{b}[0] + 2\sum_{n=0}^{(K-1)/2} \mathrm{b}[2n+1]\cos\left((2n+1)\Omega\right)$$

$$\mathrm{B}\left(e^{j\Omega}\right) = \mathrm{b}[0] + 2\left\{\mathrm{b}[1]\cos(\Omega) + \mathrm{b}[3]\cos(3\Omega) + \cdots + \mathrm{b}[K]\cos(K\Omega)\right\}$$

$$\mathrm{B}\left(e^{j(\pi-\Omega)}\right) = \mathrm{b}[0] - 2\left\{\mathrm{b}[1]\cos(\Omega) + \mathrm{b}[3]\cos(3\Omega) + \cdots + \mathrm{b}[K]\cos(K\Omega)\right\}$$

$$\mathrm{B}\left(e^{j\Omega}\right) + \mathrm{B}\left(e^{j(\pi-\Omega)}\right) = 2\,\mathrm{b}[0] \text{ a constant for all } \Omega.$$

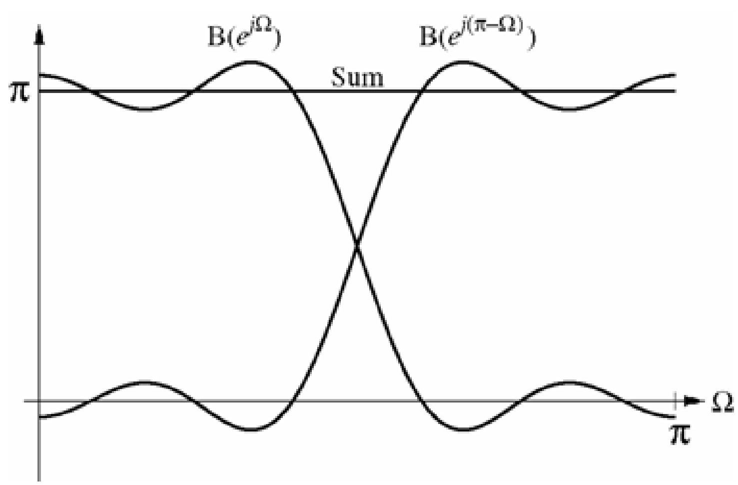
Perfect Reconstruction QMF

As an example let

$$\mathrm{b}[n] = \left\{ \begin{array}{l} \cdots,0,0,-1/7,0,1/5,0,-1/3,0,1, \\ \underset{\uparrow}{\frac{\pi}{2}},1,0,-1/3,0,1/5,0,-1/7,0,0,\cdots \end{array} \right\}$$

Then

$$\mathrm{B}\left(e^{j\Omega}\right) = \frac{\pi}{2} + 2\left\{\cos(\Omega) + \cos(3\Omega)/3 + \cos(5\Omega)/5\right\}$$



Perfect Reconstruction QMF

The filter with impulse response $\mathrm{b}[n]$ is non-causal. It can be made causal by delaying it by K samples. Also the frequency response $\mathrm{B}(e^{j\Omega})$ goes negative at some frequencies. If we add a term Δ_B just large enough to make it non-negative at all frequencies we get $\mathrm{B}_+(e^{j\Omega}) = \mathrm{B}(e^{j\Omega}) + \Delta_B$. Since it is non-negative it is possible to write it in the form

$$\mathrm{B}_+(e^{j\Omega}) = \left|\mathrm{H}(e^{j\Omega})\right|^2 = \mathrm{H}(e^{j\Omega})\mathrm{H}(e^{-j\Omega})$$

Since $\mathrm{h}[n] * \mathrm{h}[-n] \xleftrightarrow{\mathcal{F}} \mathrm{H}(e^{j\Omega})\mathrm{H}(e^{-j\Omega})$ we can say that the corresponding impulse response has a length N if $\mathrm{b}[n]$ has length $2N-1$.

Perfect Reconstruction QMF

Now delay the original impulse response by $N-1$ samples to make it causal and redefine it as

$$\mathrm{B}_{+}\left(e^{j\Omega}\right)=\left|\mathrm{H}\left(e^{j\Omega}\right)\right|^{2} e^{-j\Omega(N-1)}=\mathrm{H}\left(e^{j\Omega}\right)\mathrm{H}\left(e^{-j\Omega}\right)e^{-j\Omega(N-1)}$$

Using the fact that

$$\mathrm{B}\left(e^{j\Omega}\right)=\mathrm{b}[0]+2\left\{\begin{array}{l}\mathrm{b}[1]\cos(\Omega)+\mathrm{b}[3]\cos(3\Omega)+\cdots \\ +\mathrm{b}[K]\cos(K\Omega)\end{array}\right\}$$

it follows that

$$\mathrm{B}_{+}\left(e^{j\Omega}\right)=\left\{\mathrm{b}[0]+2\left\{\begin{array}{l}\mathrm{b}[1]\cos(\Omega)+\mathrm{b}[3]\cos(3\Omega)+\cdots \\ +\mathrm{b}[K]\cos(K\Omega)\end{array}\right\}+\Delta_{B}\right\}e^{-j\Omega(N-1)}$$

Perfect Reconstruction QMF

After a few lines of algebraic simplification we can show that

$$\mathrm{B}_+\left(e^{j\Omega}\right)+(-1)^{N-1}\mathrm{B}_+\left(e^{j(\Omega-\pi)}\right)$$

$$=\mathrm{B}_+\left(e^{j\Omega}\right)-\mathrm{B}_+\left(e^{j(\Omega-\pi)}\right)=2\,\mathrm{b}[0]\,e^{-j\Omega(N-1)}=\alpha e^{-j\Omega(N-1)}$$

where $\alpha = 2\,\mathrm{b}[0]$ a constant. It then follows that

$$\mathrm{H}(z)\mathrm{H}\left(z^{-1}\right)+\mathrm{H}(-z)\mathrm{H}\left(-z^{-1}\right)=\alpha$$

Then combining the conditions for perfect reconstruction with the conditions for elimination of aliases,

$$\mathrm{H}_0(z)=\mathrm{H}(z)$$

$$\mathrm{H}_1(z)=-z^{-(N-1)}\mathrm{H}_0\left(-z^{-1}\right)$$

$$\mathrm{G}_0(z)=\mathrm{H}_1(-z)=-(-z)^{-(N-1)}\mathrm{H}_0\left(z^{-1}\right)=z^{-(N-1)}\mathrm{H}_0\left(z^{-1}\right)$$

$$\mathrm{G}_1(z)=-\mathrm{H}_0(-z)=z^{-(N-1)}\mathrm{H}_1\left(z^{-1}\right)$$

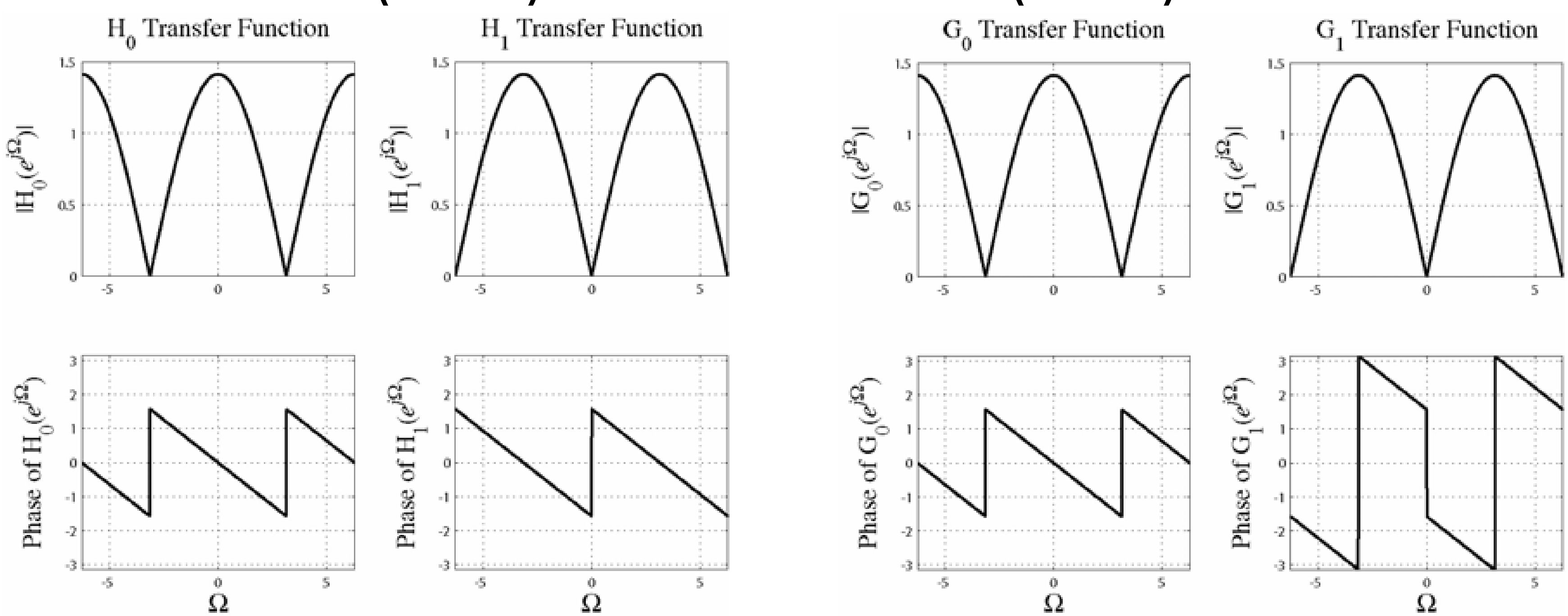
Perfect Reconstruction QMF

As an example let $H_0(z) = \left(1/\sqrt{2}\right)\left(1+z^{-1}\right)$.

Then the four filters in the perfect reconstruction QMF analysis structure are

$$H_0(z) = \left(1/\sqrt{2}\right)\left(1+z^{-1}\right) \quad H_1(z) = \left(1/\sqrt{2}\right)\left(1-z^{-1}\right)$$

$$G_0(z) = \left(1/\sqrt{2}\right)\left(1+z^{-1}\right) \quad G_1(z) = -\left(1/\sqrt{2}\right)\left(1-z^{-1}\right)$$



Perfect Reconstruction QMF

The four impulse responses are

$$h_0[n] = \left(1/\sqrt{2}\right)\left(\delta[n] + \delta[n-1]\right)$$

$$h_1[n] = \left(1/\sqrt{2}\right)\left(\delta[n] - \delta[n-1]\right)$$

$$g_0[n] = \left(1/\sqrt{2}\right)\left(\delta[n] + \delta[n-1]\right)$$

$$g_1[n] = -\left(1/\sqrt{2}\right)\left(\delta[n] - \delta[n-1]\right)$$

and the response to a random excitation is a perfect reconstruction except delayed and multiplied by two.