Parametric Spectral Estimation

ARMA Models

The most common model used in parametric spectral estimation is the rational function model used to describe ARMA systems

$$\mathrm{H}(z) = \frac{\mathrm{B}(z)}{\mathrm{A}(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}}$$

with the corresponding difference equation

$$\mathrm{x}[n] = -\sum_{k=1}^{p} a_k \, \mathrm{x}[n-k] + \sum_{k=0}^{q} b_k \, \mathrm{w}[n-k]$$

ARMA Models

The power spectral density is

$$G_{xx}(F) = G_{ww}(F)\left|H(F)\right|^2$$

If the excitation w is a zero-mean white noise process then

$$G_{xx}(F) = \sigma_{ww}^2 \left|H(F)\right|^2 = \sigma_{ww}^2 \left|\frac{B\left(e^{j2\pi F}\right)}{A\left(e^{j2\pi F}\right)}\right|^2$$

The most common form of model is the AR model in which $q = 0$ and $b_0 = 1$ because it usually results in the fewest parameters needed to represent a process. In that case

$$G_{xx}(F) = \frac{\sigma_{ww}^2}{\left|A\left(e^{j2\pi F}\right)\right|^2}$$

ARMA Models

The normal equations

$$R_{xx}\left[l\right] = -\sum_{k=1}^{p} a_p\left[k\right] R_{xx}\left[l-k\right] \ , \ \ l = 1, 2, \cdots, p$$

can be use to find the coefficients a.

ARMA Models

Example

Let the autocorrelation of the response of a system to an applied white noise signal with variance $\sigma_w^2 = 6$ have the following values

$$R_{xx}[0] = 4 \ , \ R_{xx}[1] = 2 \ , \ R_{xx}[2] = 3$$

Model the system producing this signal as an AR(2) system and graph the power spectral density of the signal.

The normal equations are

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \\ R_{xx}[1] & R_{xx}[0] \end{bmatrix} \begin{bmatrix} a_4[1] \\ a_4[2] \end{bmatrix} = \begin{bmatrix} R_{xx}[1] \\ R_{xx}[2] \end{bmatrix}$$

ARMA Models

Example

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a_4[1] \\ a_4[2] \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_4[1] \\ a_4[2] \end{bmatrix} = \begin{bmatrix} -0.1667 \\ -0.667 \end{bmatrix}$$

$$\mathrm{H}(z) = \frac{1}{\mathrm{A}(z)} = \frac{1}{1 - 0.1667 z^{-1} - 667 z^{-2}}$$

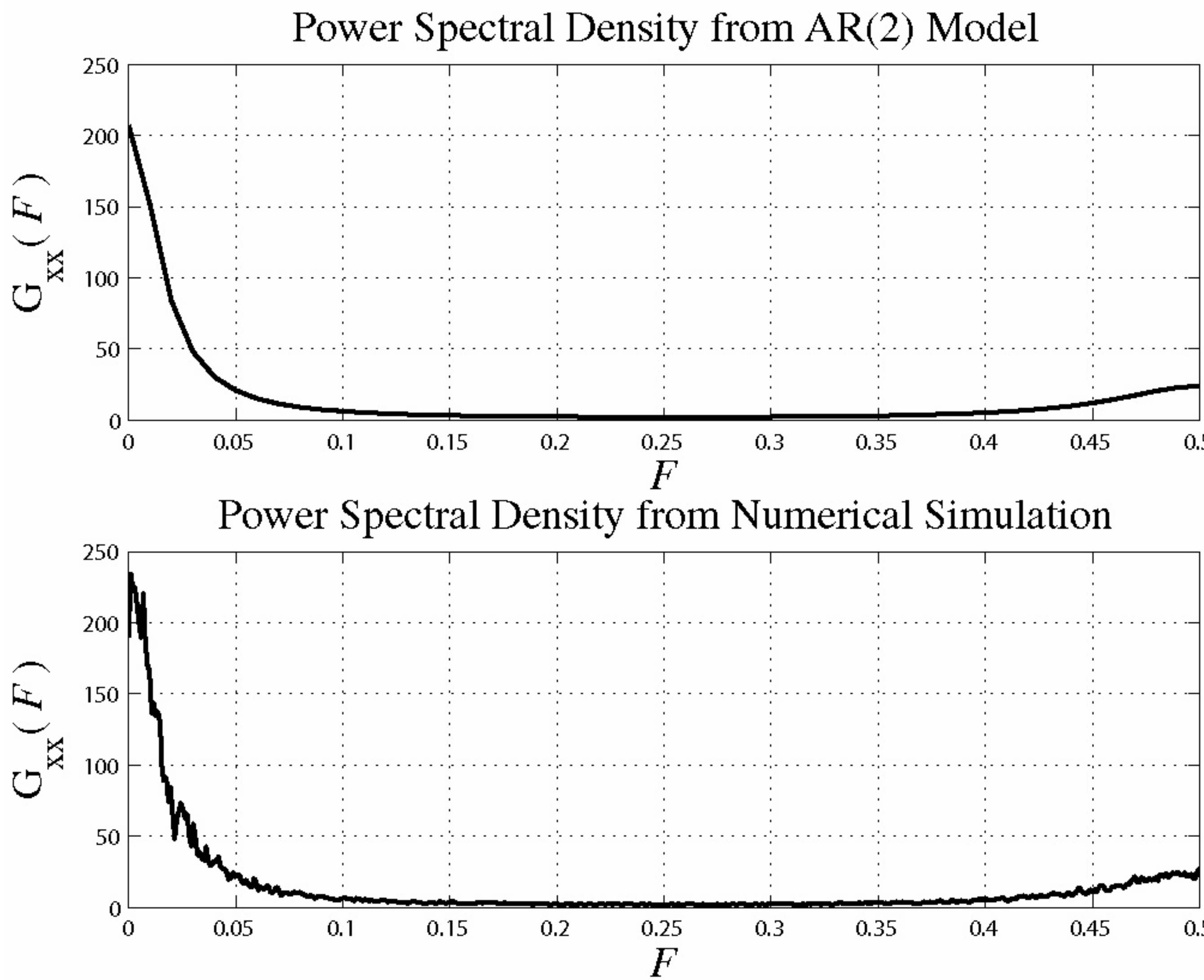
The power spectral density is

$$\mathrm{G}_{xx}(F) = \frac{6}{\left|1 - 0.1667 e^{-j2\pi F} - 0.667 e^{-j4\pi F}\right|^2}$$

$$\mathrm{G}_{xx}(F) = \frac{6}{1.473 - 0.111\cos(2\pi F) - 1.333\cos(4\pi F)}$$

ARMA Models

Example



MA Models

For an $\mathrm{MA}(q)$ system the transfer function is

$$\mathrm{H}(z) = \mathrm{B}(z) = \sum_{k=0}^{q} b_k z^{-k}$$

and the impulse response is $\mathrm{h}[n] = \sum_{k=0}^{q} b_k \delta[n-k]$. So the squared magnitude of the transfer function is

$$\mathrm{H}(z)\mathrm{H}(z^{-1}) = \mathrm{B}(z)\mathrm{B}(z^{-1}) = \mathrm{D}(z) = \sum_{k=-q}^{q} d_k z^{-k}$$

where $\sum_{k=-q}^{q} d_k \delta[n-k] = \mathrm{h}[n] * \mathrm{h}[-n]$.

MA Models

$$\left(\mathrm{h}[n] * \mathrm{h}[-n]\right)_{n=0} = b_0^2 + b_1^2 + \cdots + b_q^2$$

$$\left(\mathrm{h}[n] * \mathrm{h}[-n]\right)_{n=1} = b_0 b_1 + b_1 b_2 + \cdots + b_{q-1} b_q = \left(\mathrm{h}[n] * \mathrm{h}[-n]\right)_{n=-1}$$

$$\left(\mathrm{h}[n] * \mathrm{h}[-n]\right)_{n=2} = b_0 b_2 + b_1 b_3 + \cdots + b_{q-2} b_q = \left(\mathrm{h}[n] * \mathrm{h}[-n]\right)_{n=-2}$$

$$\vdots \qquad\qquad \vdots$$

$$\left(\mathrm{h}[n] * \mathrm{h}[-n]\right)_{n=m} = \sum_{k=0}^{q-|m|} b_k b_{k+m} \quad , \quad |m| \le q$$

MA Models

The autocorrelation of x is

$$\mathrm{R}_{xx}\left[m\right] = \mathrm{R}_{ww}\left[m\right] * \mathrm{h}\left[m\right] * \mathrm{h}\left[-m\right] = \sigma_w^2\,\mathrm{h}\left[m\right] * \mathrm{h}\left[-m\right]$$

$$\mathrm{R}_{xx}\left[m\right] = \sigma_w^2 \sum_{k=0}^{q-|m|} b_k b_{k+m} \quad , \quad |m| \le q$$

$$\mathrm{R}_{xx}\left[m\right] = \sigma_w^2 d_m$$

Therefore

$$\mathrm{G}_{xx}\left(F\right) = \sum_{m=-\infty}^{\infty} \mathrm{R}_{\mathrm{xx}}\left[m\right] e^{-j2\pi Fm} = \sum_{m=-q}^{q} \mathrm{R}_{\mathrm{xx}}\left[m\right] e^{-j2\pi Fm}$$