Realizations of Discrete-Time Systems

Direct Form II

For a discrete-time system described by

$$\sum_{k=0}^{N} a_k \, \mathrm{y}[n-k] = \sum_{k=0}^{M} b_k \, \mathrm{x}[n-k]$$

the transfer function is of the form

$$\mathrm{H}(z) = \frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_N}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

(Here the order of the numerator and denominator are both indicated as *N*. If the order of the numerator is actually less than *N*, some of the *b*'s will be zero. But a_0 must not be zero.)

Direct Form II

$$\mathrm{H}(z) = \mathrm{H}_1(z)\mathrm{H}_2(z)$$

where

$$\mathrm{H}_1(z) = \frac{\mathrm{Y}_1(z)}{\mathrm{X}(z)} = \frac{1}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

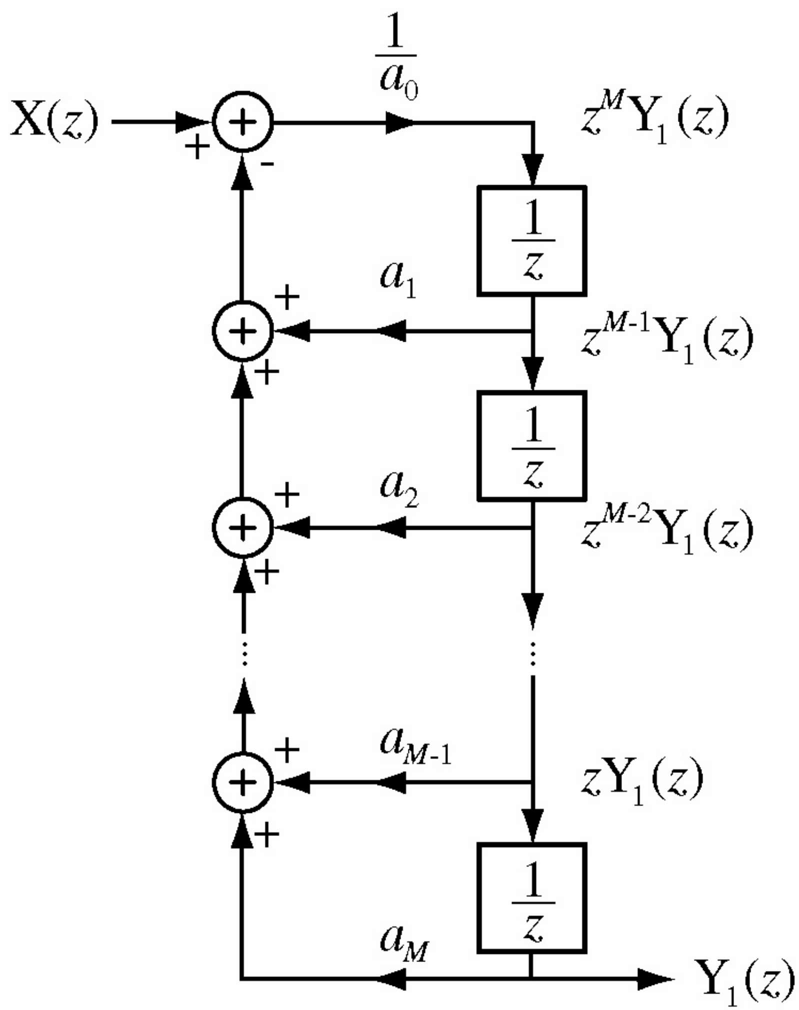
$$\mathrm{H}_2(z) = \frac{\mathrm{Y}(z)}{\mathrm{Y}_1(z)} = b_0 z^N + b_1 z^{N-1} + \cdots + b_N$$

Rearranging $\mathrm{H}_1(z)$,

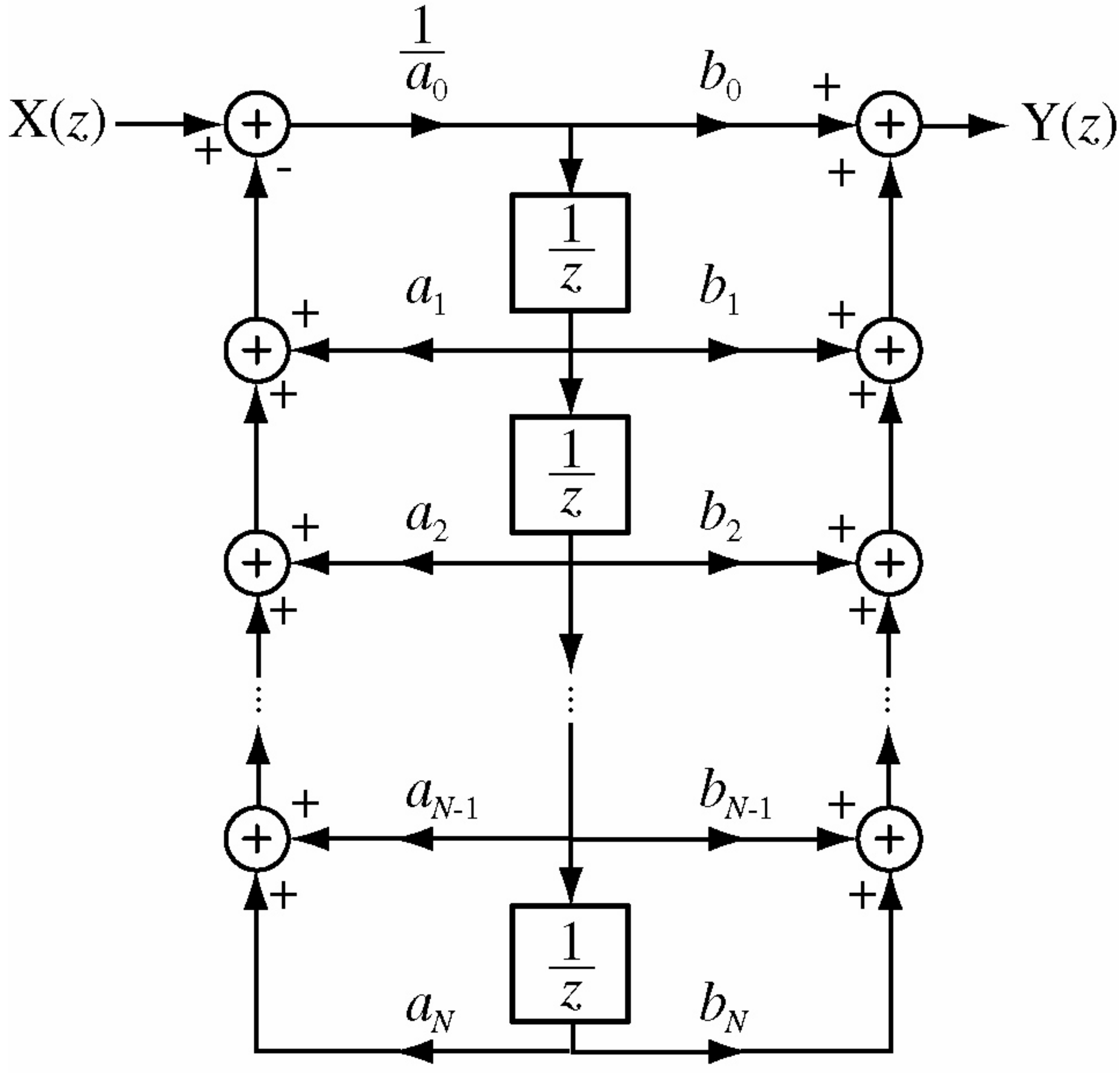
$$z^N \, \mathrm{Y}_1(z) = (1/a_0)\left\{\mathrm{X}(z) - \left[a_1 z^{N-1}\,\mathrm{Y}_1(z) + \cdots + a_N\,\mathrm{Y}_1(z)\right]\right\}$$

Direct Form II

$$z^N \mathrm{Y}_1(z) = (1/a_0)\left\{\mathrm{X}(z) - \left[a_1 z^{N-1} \mathrm{Y}_1(z) + \cdots + a_N \mathrm{Y}_1(z)\right]\right\}$$



Direct Form II

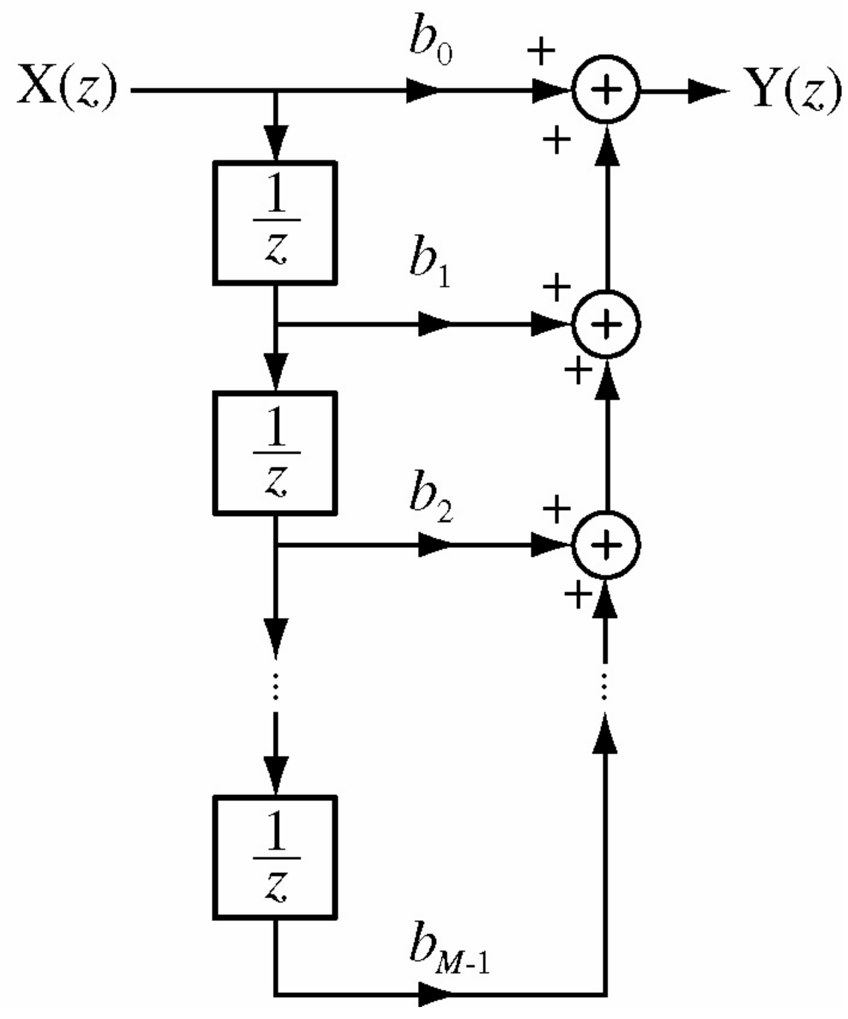


Direct Form II

For the special case of FIR filters. (The number of delays has been changed to *M* - 1 to conform to conventions in the DSP literature.)

$$\mathrm{h}[n] = \sum_{k=0}^{M-1} b_k \delta[n-k]$$

This type of filter has *M* - 1 finite zeros and *M* - 1 poles at *z* = 0.



Direct Form II

One desirable characteristic of an FIR filter is that it can have linear phase in its passband.

The impulse response is

$$\mathrm{h}[n] = \mathrm{h}[0]\delta[n] + \mathrm{h}[1]\delta[n-1] + \cdots + \mathrm{h}[M-1]\delta[n-(M-1)]$$

and its z transform is

$$\mathrm{H}(z) = \mathrm{h}[0] + \mathrm{h}[1]z^{-1} + \cdots + \mathrm{h}[M-1]z^{-(M-1)}$$

and its frequency response is

$$\mathrm{H}(e^{j\Omega}) = \mathrm{h}[0] + \mathrm{h}[1]e^{-j\Omega} + \cdots + \mathrm{h}[M-1]e^{-j(M-1)\Omega}$$

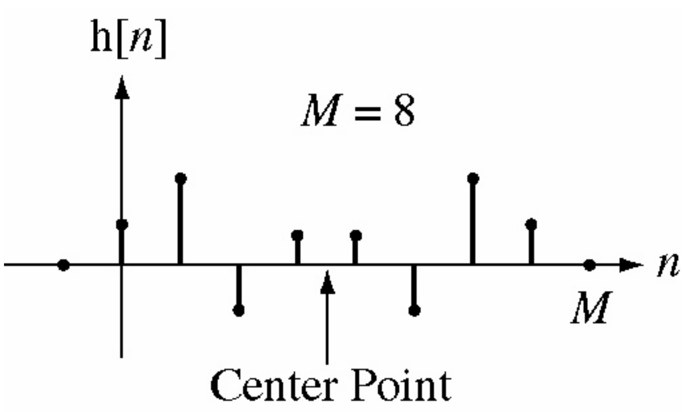
Direct Form II

The response $\mathrm{y}[n]$ to an excitation $\mathrm{x}[n]$ is

$$\mathrm{y}[n] = b_0\,\mathrm{x}[n] + b_1\,\mathrm{x}[n-1] + \cdots + b_{M-1}\,\mathrm{x}\big[n-(M-1)\big].$$

Let M be even and let the filter coefficients be chosen such that

$$\mathrm{h}[0] = \mathrm{h}[M-1]\ ,\ \ \mathrm{h}[1] = \mathrm{h}[M-2]\ ,\ \ \cdots\ ,\ \ \mathrm{h}[M/2-1] = \mathrm{h}[M/2]$$



Direct Form II

Then the frequency response is

$$H\left(e^{j\Omega}\right) = \begin{Bmatrix} h[0] + h[0]e^{-j(M-1)\Omega} + h[1]e^{-j\Omega} + h[1]e^{-j(M-2)\Omega} + \cdots \\ + h[M/2-1]e^{-j(M/2-1)\Omega} + h[M/2-1]e^{-j(M/2)\Omega} \end{Bmatrix}$$

or

$$H\left(e^{j\Omega}\right) = e^{-j\left(\frac{M-1}{2}\right)\Omega} \begin{Bmatrix} h[0]\left(e^{j((M-1)/2)\Omega} + e^{-j((M-1)/2)\Omega}\right) \\ + h[1]\left(e^{j((M-3)/2)\Omega} + e^{-j((M-3)/2)\Omega}\right) + \cdots \\ + h[M/2-1]\left(e^{-j\Omega} + e^{j\Omega}\right) \end{Bmatrix}$$

or

$$H\left(e^{j\Omega}\right) = 2e^{-j\left(\frac{M-1}{2}\right)\Omega} \begin{Bmatrix} h[0]\cos\left(\left(\frac{M-1}{2}\right)\Omega\right) + h[1]\cos\left(\left(\frac{M-3}{2}\right)\Omega\right) + \cdots \\ + h[M/2-1]\cos(\Omega) \end{Bmatrix}$$

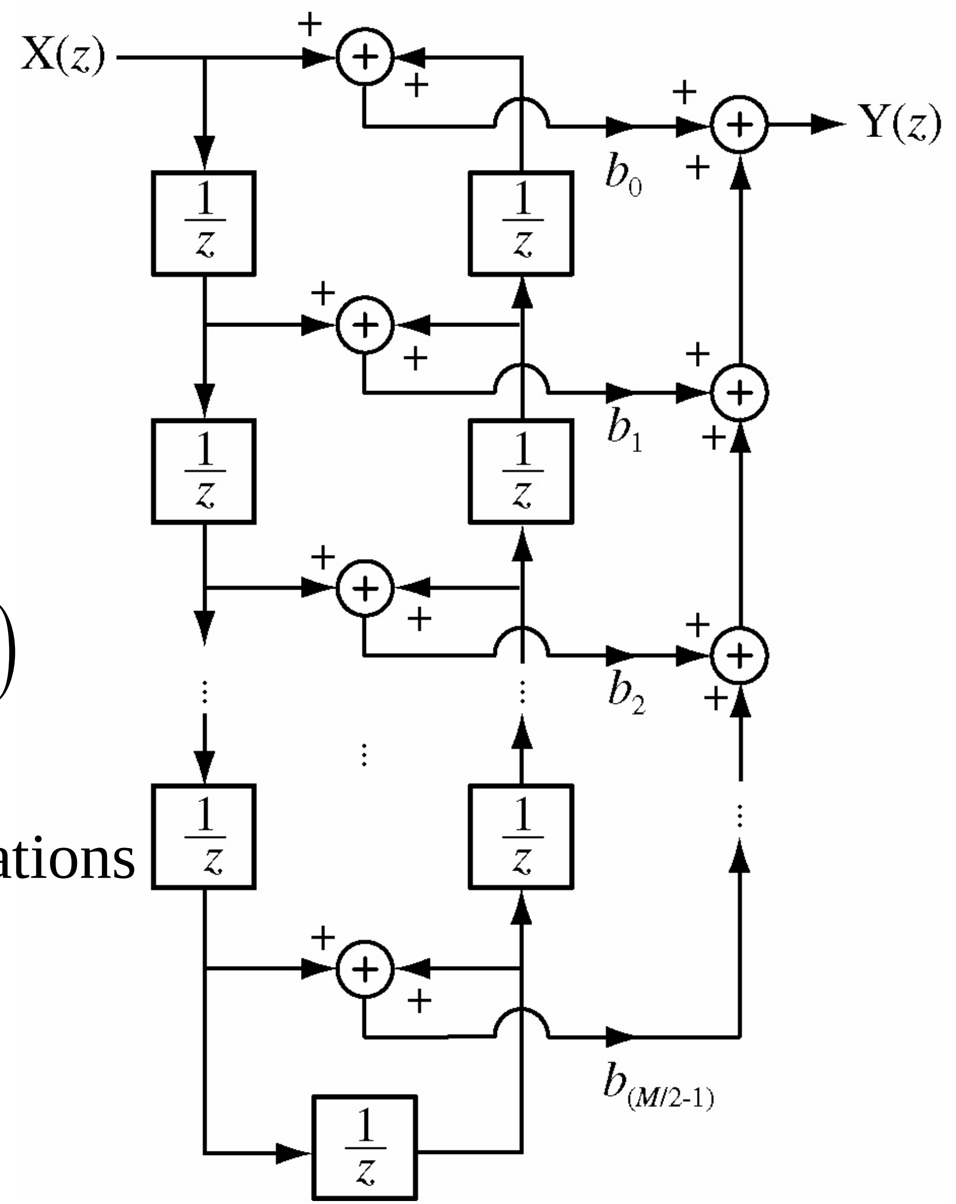
Direct Form II

In the frequency response

$$H\left(e^{j\Omega}\right) = 2e^{-j\left(\frac{M-1}{2}\right)\Omega}\left\{\begin{array}{l} h[0]\cos\left(\left(\frac{M-1}{2}\right)\Omega\right) + h[1]\cos\left(\left(\frac{M-3}{2}\right)\Omega\right) + \cdots \\ \quad + h[M/2-1]\cos(\Omega) \end{array}\right\}$$

there is a factor $e^{-j((M-1)/2)\Omega}$ which has a linear phase and the rest of the frequency response is purely real.

Direct Form II



The recursion relation is

$$y[n] = b_0\left(x[n] + x\left[n-(M-1)\right]\right)$$

$$+b_1\left(x[n-1] + x\left[n-(M-2)\right]\right) + \cdots$$

$$+b_{M/2-1}\left(x\left[n-(M/2-1) + x[M/2]\right]\right)$$

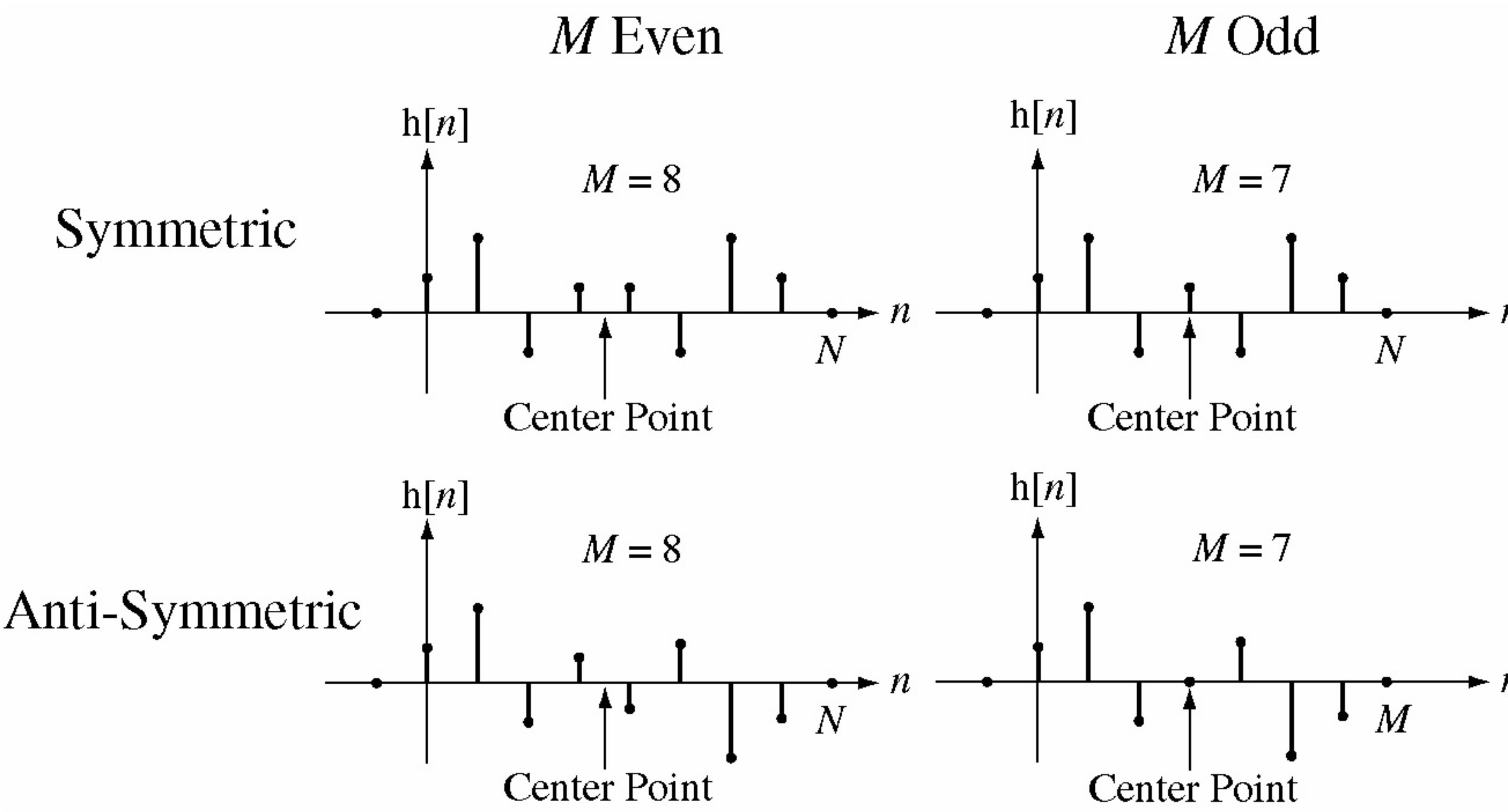
which can be realized in this form

that reduces the number of multiplications

by half.

Direct Form II

It can be shown that symmetric or anti-symmetric, even or odd impulse responses yield linear phase shift in the frequency response.



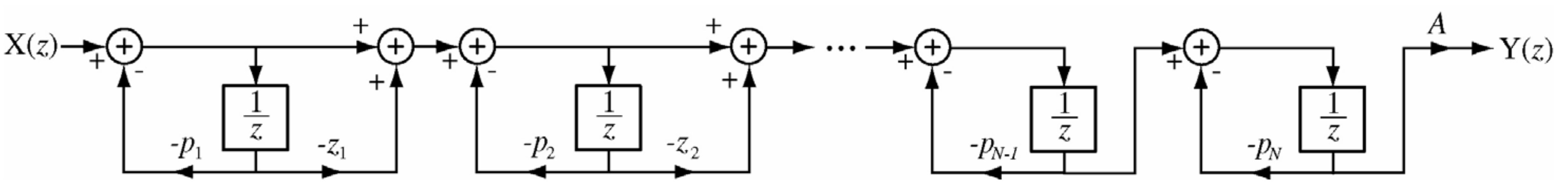
Cascade Realization

Direct Form II is not the only form of realization. There are several other forms. Two other important forms are the cascade form and the parallel form.

The cascade form is realized by first factoring the transfer function

$$H(z) = A\frac{z-z_1}{z-p_1}\frac{z-z_2}{z-p_2}\cdots\frac{z-z_M}{z-p_M}\frac{1}{z-p_{M+1}}\frac{1}{z-p_{M+2}}\cdots\frac{1}{z-p_N}$$

Each individual factor is realized as a small Direct Form II subsystem and the subsystems are then cascaded.

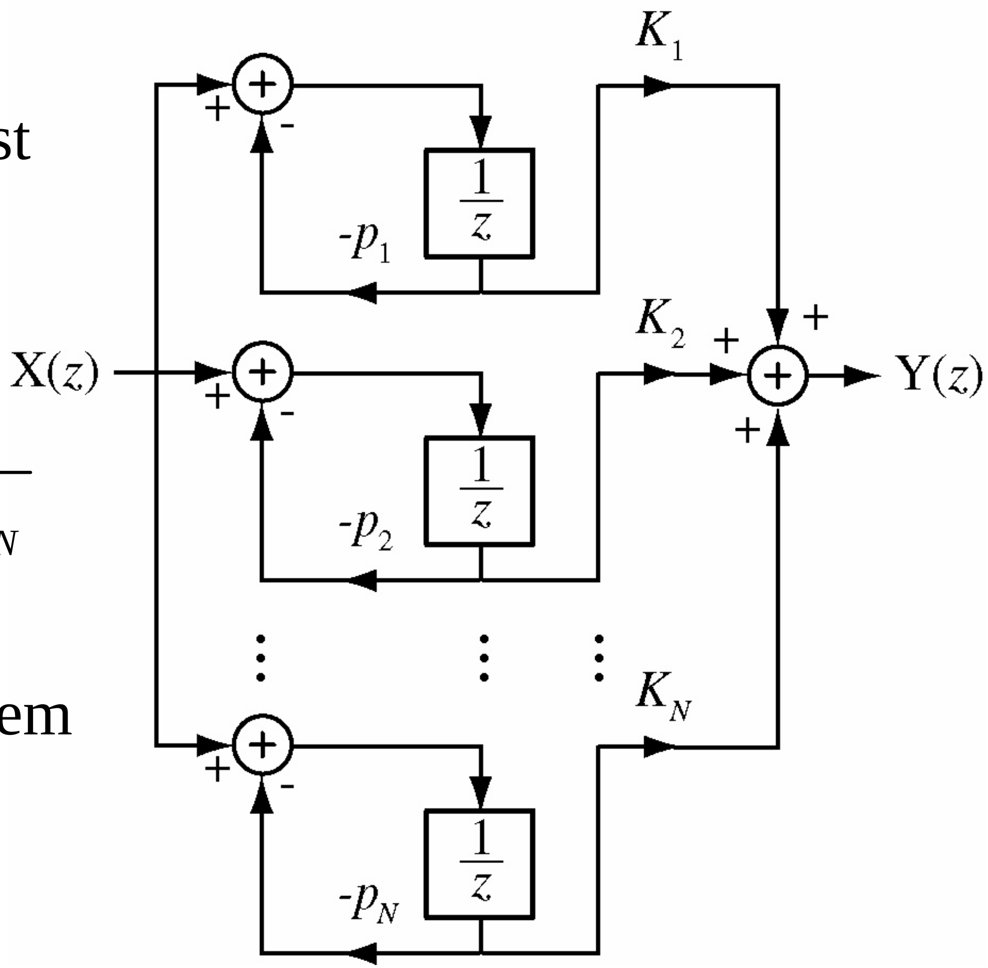


Parallel Realization

The parallel form is realized by first expressing the transfer function in partial-fraction form

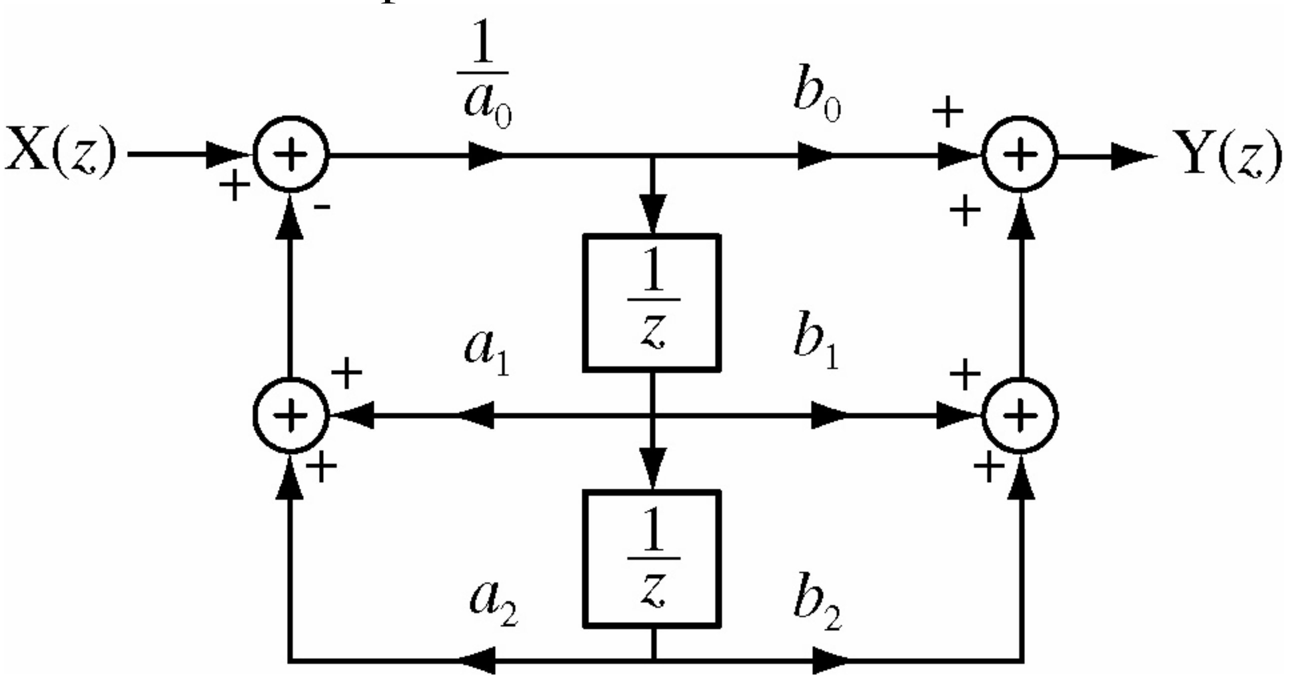
$$\mathrm{H}(z) = \frac{K_1}{z - p_1} + \frac{K_2}{z - p_2} + \cdots + \frac{K_N}{z - p_N}$$

Each individual term is realized as a small Direct Form II subsystem and the subsystems are then paralleled.



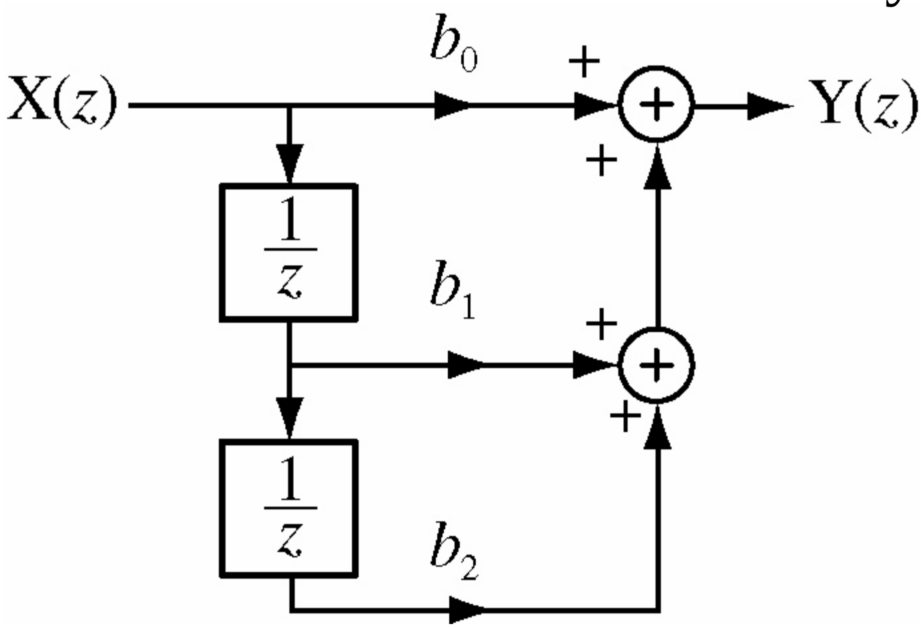
Complex Poles and Zeros

In either the cascade or parallel realization, the first-order subsystems may have complex poles and/or zeros. In such a case two first-order subsystems should be combined into one second-order subsystem to avoid the problem of complex coefficients in the first-order subsystems. Also, for reasons we will soon see, it is common to do cascade and parallel realizations with second-order subsystems even when the poles and/or zeros are real.



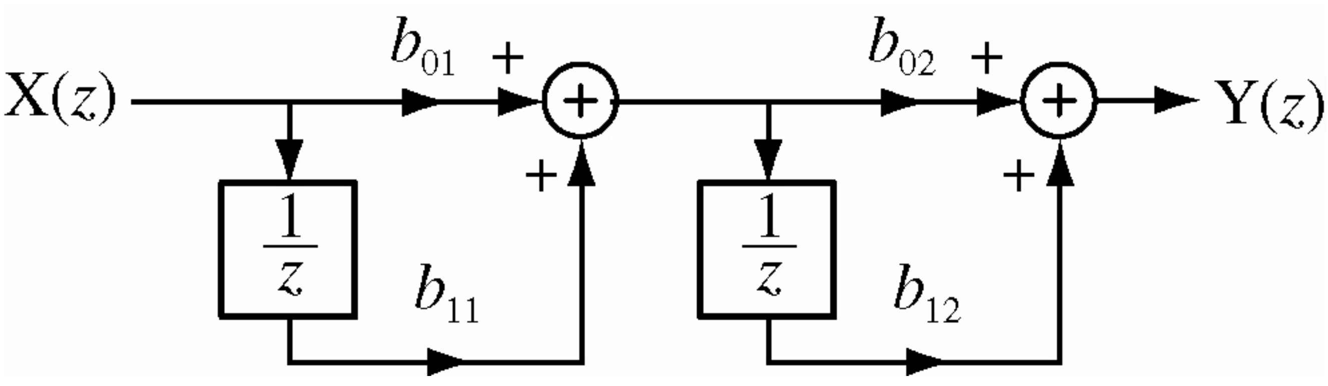
1st Order vs 2nd Order

In the case of FIR filters the second-order subsystems take this form.



2 delays
3 multiplications
2 additions

Compare this two two first-order cascaded stages.



2 delays
4 multiplications
2 additions

1st Order vs 2nd Order

If the FIR filter has linear phase, a fourth-order structure reduces number of multiplications even further compared with cascaded first-order or cascaded second-order subsystems.