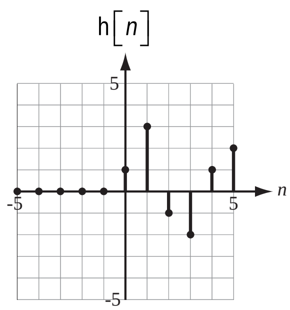
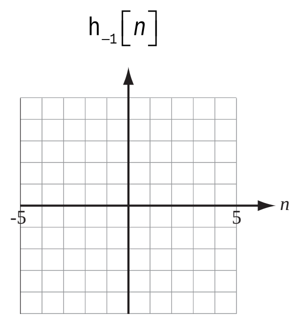
Solution of ECE 315 Test 3 Su08

1. The impulse response $\mathrm{h}[n]$ of an LTI system is illustrated below. Graph the unit sequence response $\mathrm{h}_{-1}[n]$ of that system over the same time range in the space provided to the right. Put a vertical scale on the graph so that actual numbers could be read from it. ($\mathrm{h}[n] = 0\ ,\ n < -5$).





$$\mathrm{h}_{-1}[n] = \mathrm{h}[n] * \mathrm{u}[n] = \sum_{m=-\infty}^{\infty} \mathrm{h}[m]\mathrm{u}[n-m] = \sum_{m=-\infty}^{n} \mathrm{h}[m]$$

n	–5	–4	–3	–2	–1	0	1	2	3	4	5
$\mathrm{h}_{-1}[n]$	0	0	0	0	0	1	4	3	1	2	4

2. An LTI system has an impulse response $\mathrm{h}(t) = 2e^{-3t}\,\mathrm{u}(t)$.

(a) Write an expression for $\mathrm{h}(t) * \mathrm{u}(t)$. (Any function convolved with a unit step is just the integral of the function, $\mathrm{h}(t) * \mathrm{u}(t) = \int_{-\infty}^{t} \mathrm{h}(\tau)\,d\tau$.)

(b) Let the excitation of the system be $\mathrm{x}(t) = \mathrm{u}(t) - \mathrm{u}(t - 1/3)$. Write an expression for the response $\mathrm{y}(t)$.

(c) Find the numerical value of $\mathrm{y}(t)$ at $t = 1/2$.

$$\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t)$$

$$\mathrm{h}(t) * \mathrm{u}(t) = \int_{-\infty}^{\infty} \mathrm{h}(\tau)\mathrm{u}(t-\tau)\,d\tau = 2\int_{-\infty}^{\infty} e^{-3\tau}\,\mathrm{u}(\tau)\mathrm{u}(t-\tau)\,d\tau$$

$$\mathrm{h}(t) * \mathrm{u}(t) = \begin{cases} 2\int_{0}^{t} e^{-3\tau}\,d\tau & , \ t \ge 0 \\ 0 & , \ t < 0 \end{cases}$$

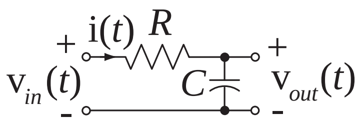
$$\mathrm{h}(t) * \mathrm{u}(t) = (2/3)\left(1 - e^{-3t}\right)\mathrm{u}(t)$$

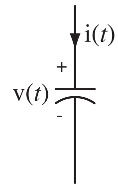
$$\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t) = \mathrm{h}(t) * \mathrm{u}(t) - \mathrm{h}(t) * \mathrm{u}(t - 1/3) = (2/3)\left(1 - e^{-3t}\right)\mathrm{u}(t) - (2/3)\left(1 - e^{-3(t-1/3)}\right)\mathrm{u}(t - 1/3)$$

$$\mathrm{y}(t) = (2/3)\left[\left(1 - e^{-3t}\right)\mathrm{u}(t) - \left(1 - e^{-3(t-1/3)}\right)\mathrm{u}(t - 1/3)\right]$$

$$\mathrm{y}(1/2) = (2/3)\left[\left(1 - e^{-3/2}\right)\mathrm{u}(1/2) - \left(1 - e^{-1/2}\right)\mathrm{u}(1/6)\right] = (2/3)\left(e^{-1/2} - e^{-3/2}\right) = 0.2556$$

3. Below is an RC lowpass filter with excitation $v_{in}(t)$ and response $v_{out}(t)$. Let $R = 10\Omega$ and $C = 10\mu F$.





$$i(t) = C\frac{d}{dt}v(t)$$

(a) Write the differential equation for this circuit in terms of $v_{in}(t)$, $v_{out}(t)$, R and C.

$$i(t) = C\frac{d}{dt}v_{out}(t) \quad \text{and} \quad v_{out}(t) + R\,i(t) = v_{in}(t)$$

Combining expressions,

$$RC\,v'_{out}(t) + v_{out}(t) = v_{in}(t)$$

$$v'_{out}(t) + \frac{v_{out}(t)}{RC} = \frac{v_{in}(t)}{RC}$$

(b) Find the impulse response of this system $h(t)$ ($h(t) = v_{out}(t)$ when $v_{in}(t) = \delta(t)$).

$$h'(t) + \frac{h(t)}{RC} = \frac{\delta(t)}{RC}$$

$$h(t) = Ke^{-t/RC}u(t)$$

$$h(0^+) - h(0^-) + \frac{1}{RC}\int_{0^-}^{0^+} h(t)\,dt = \frac{1}{RC}\left[u(0^+) - u(0^-)\right]$$

$$K = \frac{1}{RC}$$

$$h(t) = \frac{e^{-t/RC}}{RC}u(t)$$

(c) Find the numerical value $h(2\times10^{-4})$.

$$h(2\times10^{-4}) = \frac{e^{-2\times10^{-4}/10^{-4}}}{10^{-4}}u(2\times10^{-4}) = \frac{e^{-2}}{10^{-4}} = 1353.35$$

4. Find these numerical values.

(a) $\mathrm{g}(3)$ if $\mathrm{g}(t) = e^{-t}\,\mathrm{u}(t) * [\delta(t) - 2\delta(t-1)]$

$$\mathrm{g}(t) = e^{-t}\,\mathrm{u}(t) - 2e^{-(t-1)}\,\mathrm{u}(t-1) \Rightarrow \mathrm{g}(3) = e^{-3} - 2e^{-2} = -0.2209$$

(b) $\mathrm{g}[3]$ if $\mathrm{g}[n] = \mathrm{ramp}[n] * \mathrm{u}[n]$

$$\mathrm{g}[n] = \sum_{m=-\infty}^{\infty} \mathrm{ramp}[m]\,\mathrm{u}[n-m] = \begin{cases} \sum_{m=0}^{n} \mathrm{ramp}[m] & , \; n \geq 0 \\ 0 & , \; n < 0 \end{cases}$$

$$\mathrm{g}[3] = \sum_{m=0}^{3} \mathrm{ramp}[m] = \mathrm{ramp}[0] + \mathrm{ramp}[1] + \mathrm{ramp}[2] + \mathrm{ramp}[3]$$

$$\mathrm{g}[3] = 0 + 1 + 2 + 3 = 6$$

(c) $\mathrm{g}[13]$ if $\mathrm{g}[n] = (\mathrm{u}[n] - \mathrm{u}[n-5]) * \delta_2[n]$

$$\mathrm{g}[n] = (\mathrm{u}[n] - \mathrm{u}[n-5]) * \sum_{k=-\infty}^{\infty} \delta[n-2k] = \sum_{k=-\infty}^{\infty} \mathrm{u}[n] * \delta[n-2k] - \sum_{k=-\infty}^{\infty} \mathrm{u}[n-5] * \delta[n-2k]$$

$$\mathrm{g}[n] = \sum_{k=-\infty}^{\infty} \mathrm{u}[n-2k] - \sum_{k=-\infty}^{\infty} \mathrm{u}[n-2k-5]$$

$$\mathrm{g}[13] = \sum_{k=-\infty}^{\infty} \mathrm{u}[13-2k] - \sum_{k=-\infty}^{\infty} \mathrm{u}[13-2k-5]$$

$$\mathrm{g}[13] = \sum_{k=-\infty}^{6} \mathrm{u}[13-2k] - \sum_{k=-\infty}^{4} \mathrm{u}[13-2k-5]$$

The summations cancel up to $k = 4$. Therefore all that is left is $k = 5,6$ for which the first summation is 2 and the second summation is 0. Therefore $\mathrm{g}[13] = 2$.

Alternate Solution:
$(\mathrm{u}[n] - \mathrm{u}[n-5])$ is a pulse starting at $n = 0$ and consisting of 5 unit impulses. When convolved with $\delta_2[n]$ we get this pulse starting at every integer multiple of 2. So at $n = 13$ we have the sum of the unit impulse from the pulse that started at $n = 10$ and the unit impulse from the pulse that started at $n = 12$. The ones that start before $n = 10$ and after $n = 12$ don't affect the value at $n = 13$. Therefore the answer is 2.

(d) The signal power of $\mathrm{g}(t) = 5\,\mathrm{rect}(8t) * \delta_4(t)$.

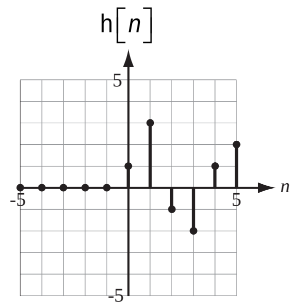
$\mathrm{g}(t) = 5\,\mathrm{rect}(8t) * \delta_4(t)$ is a periodic sequence of rectangular pulses centered at every integer multiple of 4 in time. The signal power is the average of the square of the magnitude of the signal. The square of the magnitude of the signal is

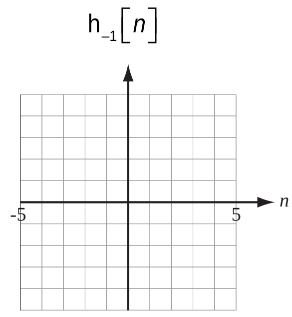
$$|\mathrm{g}(t)|^2 = 25\,\mathrm{rect}^2(8t) * \delta_4(t)$$

Its period is 4 and its area during one period is 25/8. Therefore its average value is 25/32 or 0.78125.

Solution of ECE 315 Test 3 Su08

1. The impulse response $\mathrm{h}[n]$ of an LTI system is illustrated below. Graph the unit sequence response $\mathrm{h}_{-1}[n]$ of that system over the same time range in the space provided to the right. Put a vertical scale on the graph so that actual numbers could be read from it. ($\mathrm{h}[n] = 0$, $n < -5$).





$$\mathrm{h}_{-1}[n] = \mathrm{h}[n] * \mathrm{u}[n] = \sum_{m=-\infty}^{\infty} \mathrm{h}[m]\mathrm{u}[n-m] = \sum_{m=-\infty}^{n} \mathrm{h}[m]$$

n	–5	–4	–3	–2	–1	0	1	2	3	4	5
$\mathrm{h}_{-1}[n]$	0	0	0	0	0	1	4	3	1	2	4

2. An LTI system has an impulse response $h(t) = 5e^{-2t}u(t)$.

(a) Write an expression for $h(t) * u(t)$. (Any function convolved with a unit step is just the integral of the function, $h(t) * u(t) = \int_{-\infty}^{t} h(\tau)d\tau$.)

(b) Let the excitation of the system be $x(t) = u(t) - u(t - 1/3)$. Write an expression for the response $y(t)$.

(c) Find the numerical value of $y(t)$ at $t = 1/2$.

$$y(t) = h(t) * x(t)$$

$$h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = 5\int_{-\infty}^{\infty} e^{-2\tau}u(\tau)u(t-\tau)d\tau$$

$$h(t) * u(t) = \begin{cases} 5\int_{0}^{t} e^{-2\tau}d\tau & , \ t \ge 0 \\ 0 & , \ t < 0 \end{cases}$$

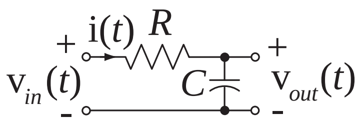
$$h(t) * u(t) = (5/2)(1 - e^{-2t})u(t)$$

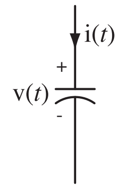
$$y(t) = h(t) * x(t) = h(t) * u(t) - h(t) * u(t - 1/3) = (5/2)(1 - e^{-2t})u(t) - (5/2)(1 - e^{-2(t-1/3)})u(t - 1/3)$$

$$y(t) = (5/2)\left[(1 - e^{-2t})u(t) - (1 - e^{-2(t-1/3)})u(t - 1/3)\right]$$

$$y(1/2) = (5/2)\left[(1 - e^{-1})u(1/2) - (1 - e^{-1/3})u(1/6)\right] = (5/2)(e^{-1/3} - e^{-1}) = 0.8717$$

3. Below is an *RC* lowpass filter with excitation $v_{in}(t)$ and response $v_{out}(t)$. Let $R = 20\Omega$ and $C = 10\mu F$.





$$i(t) = C\frac{d}{dt}v(t)$$

(a) Write the differential equation for this circuit in terms of $v_{in}(t)$, $v_{out}(t)$, *R* and *C*.

$$i(t) = C\frac{d}{dt}v_{out}(t) \quad \text{and} \quad v_{out}(t) + R\,i(t) = v_{in}(t)$$

Combining expressions,

$$RC\,v'_{out}(t) + v_{out}(t) = v_{in}(t)$$

$$v'_{out}(t) + \frac{v_{out}(t)}{RC} = \frac{v_{in}(t)}{RC}$$

(b) Find the impulse response of this system $h(t)$ ($h(t) = v_{out}(t)$ when $v_{in}(t) = \delta(t)$).

$$h'(t) + \frac{h(t)}{RC} = \frac{\delta(t)}{RC}$$

$$h(t) = Ke^{-t/RC}\,u(t)$$

$$h(0^+) - h(0^-) + \frac{1}{RC}\int_{0^-}^{0^+} h(t)\,dt = \frac{1}{RC}\left[u(0^+) - u(0^-)\right]$$

$$K = \frac{1}{RC}$$

$$h(t) = \frac{e^{-t/RC}}{RC}u(t)$$

(c) Find the numerical value $h(2\times 10^{-4})$.

$$h(2\times 10^{-4}) = \frac{e^{-2\times 10^{-4}/2\times 10^{-4}}}{2\times 10^{-4}}u(2\times 10^{-4}) = \frac{e^{-1}}{2\times 10^{-4}} = 1839.4$$

4. Find these numerical values.

(a) $g(4)$ if $g(t) = e^{-t}\,u(t) * [\delta(t) - 2\delta(t-1)]$

$$g(t) = e^{-t}\,u(t) - 2e^{-(t-1)}\,u(t-1) \Rightarrow g(4) = e^{-4} - 2e^{-3} = -0.0813$$

(b) $g[4]$ if $g[n] = \text{ramp}[n] * u[n]$

$$g[n] = \sum_{m=-\infty}^{\infty} \text{ramp}[m]\,u[n-m] = \begin{cases} \sum_{m=0}^{n} \text{ramp}[m] & , \; n \geq 0 \\ 0 & , \; n < 0 \end{cases}$$

$$g[4] = \sum_{m=0}^{4} \text{ramp}[m] = \text{ramp}[0] + \text{ramp}[1] + \text{ramp}[2] + \text{ramp}[3] + \text{ramp}[4]$$

$$g[3] = 0 + 1 + 2 + 3 + 4 = 10$$

(c) $g[14]$ if $g[n] = (u[n] - u[n-5]) * \delta_2[n]$

$$g[n] = (u[n] - u[n-5]) * \sum_{k=-\infty}^{\infty} \delta[n-2k] = \sum_{k=-\infty}^{\infty} u[n] * \delta[n-2k] - \sum_{k=-\infty}^{\infty} u[n-5] * \delta[n-2k]$$

$$g[n] = \sum_{k=-\infty}^{\infty} u[n-2k] - \sum_{k=-\infty}^{\infty} u[n-2k-5]$$

$$g[14] = \sum_{k=-\infty}^{\infty} u[14-2k] - \sum_{k=-\infty}^{\infty} u[14-2k-5]$$

$$g[13] = \sum_{k=-\infty}^{7} u[14-2k] - \sum_{k=-\infty}^{4} u[14-2k-5]$$

The summations cancel up to $k = 4$. Therefore all that is left is $k = 5,6,7$ for which the first summation is 3 and the second summation is 0. Therefore $g[14] = 3$.

Alternate Solution:
$(u[n] - u[n-5])$ is a pulse starting at $n = 0$ and consisting of 5 unit impulses. When convolved with $\delta_2[n]$ we get this pulse starting at every integer multiple of 2. So at $n = 14$ we have the sum of the unit impulse from the pulse that started at $n = 10$ and the unit impulse from the pulse that started at $n = 12$ and the unit impulse from the pulse that started at $n = 14$. The ones that start before $n = 10$ and after $n = 14$ don't affect the value at $n = 14$. Therefore the answer is 3.

(d) The signal power of $g(t) = 5\,\text{rect}(6t) * \delta_4(t)$.

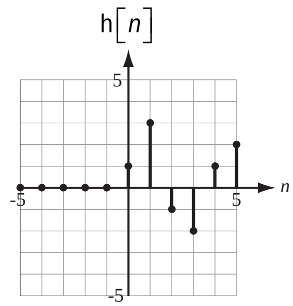
$g(t) = 5\,\text{rect}(6t) * \delta_4(t)$ is a periodic sequence of rectangular pulses centered at every integer multiple of 4 in time. The signal power is the average of the square of the magnitude of the signal. The square of the magnitude of the signal is

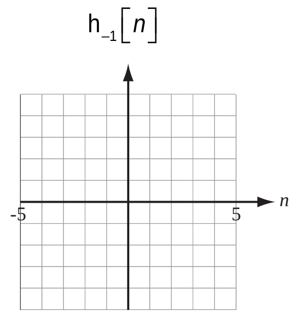
$$|g(t)|^2 = 25\,\text{rect}^2(6t) * \delta_4(t)$$

Its period is 4 and its area during one period is 25/6. Therefore its average value is 25/24 or 1.04167

Solution of ECE 315 Test 3 Su08

1. The impulse response $\mathrm{h}[n]$ of an LTI system is illustrated below. Graph the unit sequence response $\mathrm{h}_{-1}[n]$ of that system over the same time range in the space provided to the right. Put a vertical scale on the graph so that actual numbers could be read from it. ($\mathrm{h}[n] = 0$, $n < -5$).





$$\mathrm{h}_{-1}[n] = \mathrm{h}[n] * \mathrm{u}[n] = \sum_{m=-\infty}^{\infty} \mathrm{h}[m]\mathrm{u}[n-m] = \sum_{m=-\infty}^{n} \mathrm{h}[m]$$

n	–5	–4	–3	–2	–1	0	1	2	3	4	5
$\mathrm{h}_{-1}[n]$	0	0	0	0	0	1	4	3	1	2	4

2. An LTI system has an impulse response $\mathrm{h}(t) = 4e^{-t}\,\mathrm{u}(t)$.

(a) Write an expression for $\mathrm{h}(t) * \mathrm{u}(t)$. (Any function convolved with a unit step is just the integral of the function, $\mathrm{h}(t) * \mathrm{u}(t) = \int_{-\infty}^{t} \mathrm{h}(\tau)\,d\tau$.)

(b) Let the excitation of the system be $\mathrm{x}(t) = \mathrm{u}(t) - \mathrm{u}(t - 1/3)$. Write an expression for the response $\mathrm{y}(t)$.

(c) Find the numerical value of $\mathrm{y}(t)$ at $t = 1/2$.

$$\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t)$$

$$\mathrm{h}(t) * \mathrm{u}(t) = \int_{-\infty}^{\infty} \mathrm{h}(\tau)\,\mathrm{u}(t-\tau)\,d\tau = 4\int_{-\infty}^{\infty} e^{-\tau}\,\mathrm{u}(\tau)\,\mathrm{u}(t-\tau)\,d\tau$$

$$\mathrm{h}(t) * \mathrm{u}(t) = \begin{cases} 4\int_{0}^{t} e^{-\tau}\,d\tau & , \ t \geq 0 \\ 0 & , \ t < 0 \end{cases}$$

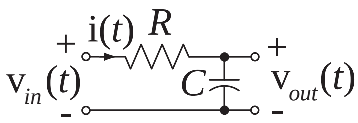
$$\mathrm{h}(t) * \mathrm{u}(t) = 4\left(1 - e^{-t}\right)\mathrm{u}(t)$$

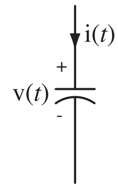
$$\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t) = \mathrm{h}(t) * \mathrm{u}(t) - \mathrm{h}(t) * \mathrm{u}(t - 1/3) = 4\left(1 - e^{-t}\right)\mathrm{u}(t) - 4\left(1 - e^{-(t-1/3)}\right)\mathrm{u}(t - 1/3)$$

$$\mathrm{y}(t) = 4\left[\left(1 - e^{-t}\right)\mathrm{u}(t) - \left(1 - e^{-(t-1/3)}\right)\mathrm{u}(t - 1/3)\right]$$

$$\mathrm{y}(1/2) = 4\left[\left(1 - e^{-1/2}\right)\mathrm{u}(1/2) - \left(1 - e^{-1/6}\right)\mathrm{u}(1/6)\right] = 4\left(e^{-1/6} - e^{-1/2}\right) = 0.9598$$

3. Below is an *RC* lowpass filter with excitation $v_{in}(t)$ and response $v_{out}(t)$. Let $R = 50\Omega$ and $C = 10\mu F$.





$$i(t) = C\frac{d}{dt}v(t)$$

(a) Write the differential equation for this circuit in terms of $v_{in}(t)$, $v_{out}(t)$, *R* and *C*.

$$i(t) = C\frac{d}{dt}v_{out}(t) \quad \text{and} \quad v_{out}(t) + R\,i(t) = v_{in}(t)$$

Combining expressions,

$$RC\,v'_{out}(t) + v_{out}(t) = v_{in}(t)$$

$$v'_{out}(t) + \frac{v_{out}(t)}{RC} = \frac{v_{in}(t)}{RC}$$

(b) Find the impulse response of this system $h(t)$ ($h(t) = v_{out}(t)$ when $v_{in}(t) = \delta(t)$).

$$h'(t) + \frac{h(t)}{RC} = \frac{\delta(t)}{RC}$$

$$h(t) = Ke^{-t/RC}\,u(t)$$

$$h(0^+) - h(0^-) + \frac{1}{RC}\int_{0^-}^{0^+} h(t)\,dt = \frac{1}{RC}\left[u(0^+) - u(0^-)\right]$$

$$K = \frac{1}{RC}$$

$$h(t) = \frac{e^{-t/RC}}{RC}\,u(t)$$

(c) Find the numerical value $h(2\times10^{-4})$.

$$h(2\times10^{-4}) = \frac{e^{-2\times10^{-4}/5\times10^{-4}}}{5\times10^{-4}}\,u(2\times10^{-4}) = \frac{e^{-2/5}}{5\times10^{-4}} = 1340.64$$

4. Find these numerical values.

(a) $g(2)$ if $g(t) = e^{-t}u(t) * [\delta(t) - 2\delta(t-1)]$

$$g(t) = e^{-t}u(t) - 2e^{-(t-1)}u(t-1) \Rightarrow g(2) = e^{-2} - 2e^{-1} = -0.6004$$

(b) $g[3]$ if $g[n] = \text{ramp}[n] * u[n]$

$$g[n] = \sum_{m=-\infty}^{\infty} \text{ramp}[m]u[n-m] = \begin{cases} \sum_{m=0}^{n} \text{ramp}[m] & , n \ge 0 \\ 0 & , n < 0 \end{cases}$$

$$g[3] = \sum_{m=0}^{3} \text{ramp}[m] = \text{ramp}[0] + \text{ramp}[1] + \text{ramp}[2] + \text{ramp}[3]$$

$$g[3] = 0 + 1 + 2 + 3 = 6$$

(c) $g[13]$ if $g[n] = (u[n] - u[n-4]) * \delta_2[n]$

$$g[n] = (u[n] - u[n-4]) * \sum_{k=-\infty}^{\infty} \delta[n-2k] = \sum_{k=-\infty}^{\infty} u[n] * \delta[n-2k] - \sum_{k=-\infty}^{\infty} u[n-4] * \delta[n-2k]$$

$$g[n] = \sum_{k=-\infty}^{\infty} u[n-2k] - \sum_{k=-\infty}^{\infty} u[n-2k-4]$$

$$g[13] = \sum_{k=-\infty}^{\infty} u[13-2k] - \sum_{k=-\infty}^{\infty} u[13-2k-4]$$

$$g[13] = \sum_{k=-\infty}^{6} u[13-2k] - \sum_{k=-\infty}^{4} u[13-2k-4]$$

The summations cancel up to $k = 4$. Therefore all that is left is $k = 5,6$ for which the first summation is 2 and the second summation is 0. Therefore $g[13] = 2$.

Alternate Solution:
$(u[n] - u[n-5])$ is a pulse starting at $n = 0$ and consisting of 5 unit impulses. When convolved with $\delta_2[n]$ we get this pulse starting at every integer multiple of 2. So at $n = 13$ we have the sum of the unit impulse from the pulse that started at $n = 10$ and the unit impulse from the pulse that started at $n = 12$. The ones that start before $n = 10$ and after $n = 12$ don't affect the value at $n = 13$. Therefore the answer is 2.

(d) The signal power of $g(t) = 5\,\text{rect}(2t) * \delta_4(t)$.

$g(t) = 5\,\text{rect}(2t) * \delta_4(t)$ is a periodic sequence of rectangular pulses centered at every integer multiple of 4 in time. The signal power is the average of the square of the magnitude of the signal. The square of the magnitude of the signal is

$$|g(t)|^2 = 25\,\text{rect}^2(2t) * \delta_4(t)$$

Its period is 4 and its area during one period is 25/2. Therefore its average value is 25/8 or 3.125.