Solution of ECE 315 Final Examination Su08

1. Find the numerical values of the constants,

(a) $A\sin\left(200\pi t+\pi/3\right)\xleftrightarrow{\mathcal{F}} j8\left[\delta\left(f+a\right)-\delta\left(f-a\right)\right]e^{bf}$

$$\sin\left(2\pi f_0 t\right)\xleftrightarrow{\mathcal{F}}\frac{j}{2}\left[\delta\left(f+f_0\right)-\delta\left(f-f_0\right)\right]$$

Multiply both sides by 16. $16\sin\left(2\pi f_0 t\right)\xleftrightarrow{\mathcal{F}} j8\left[\delta\left(f+f_0\right)-\delta\left(f-f_0\right)\right]$

Identify $f_0=100$. $16\sin\left(200\pi t\right)\xleftrightarrow{\mathcal{F}} j8\left[\delta\left(f+100\right)-\delta\left(f-100\right)\right]$

Use the time-shifting property.

$$16\sin\left(200\pi\left(t-t_0\right)\right)\xleftrightarrow{\mathcal{F}} j8\left[\delta\left(f+100\right)-\delta\left(f-100\right)\right]e^{-j2\pi f t_0}$$

$$A=16\ ,\ a=100$$

Solve for t_0.

$$16\sin\left(200\pi t-200\pi t_0\right)\xleftrightarrow{\mathcal{F}} j8\left[\delta\left(f+100\right)-\delta\left(f-100\right)\right]e^{-j2\pi f t_0}$$

$$-200\pi t_0=\pi/3$$

$$t_0=-\frac{\pi/3}{200\pi}=-\frac{1}{600}\Rightarrow b=-j2\pi t_0=\left(-j\right)\left(-\frac{2\pi}{600}\right)=j\frac{\pi}{300}\ \text{ or }\ j0.01047$$

(b) $A\,\text{rect}\left(t/a\right)*\text{rect}\left(t/b\right)\xleftrightarrow{\mathcal{F}}40\,\text{sinc}\left(2f\right)\text{sinc}\left(4f\right)$

$$A\,\text{rect}\left(t/a\right)*\text{rect}\left(t/b\right)\xleftrightarrow{\mathcal{F}}A\times a\,\text{sinc}\left(af\right)\times b\,\text{sinc}\left(bf\right)=40\,\text{sinc}\left(2f\right)\text{sinc}\left(4f\right)$$

$A\times a\times b=40$		$A\times a\times b=40$
$a=2$	or	$a=4$
$b=4\Rightarrow A=40/8=5$		$b=2\Rightarrow A=40/8=5$

Either set of answers is correct.

(c) $\frac{d}{dt}\left(3\,\text{tri}\left(5t\right)\right)\xleftrightarrow{\mathcal{F}}Af\,\text{sinc}^2\left(af\right)$

$$\frac{d}{dt}\left(3\,\text{tri}\left(5t\right)\right)\xleftrightarrow{\mathcal{F}}j2\pi f\times\left(3/5\right)\text{sinc}^2\left(f/5\right)=Af\,\text{sinc}^2\left(af\right)$$

$$A=j2\pi\times\left(3/5\right)=j6\pi/5=j3.77$$
$$a=1/5$$

2. Find the numerical values of the constants.

(a) $$10\cos\left(\frac{5\pi n}{14}\right) \xleftrightarrow{\mathcal{F}} A\left[\delta_1(F-a)+\delta_1(F+a)\right]$$

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta_1(F-F_0)+\delta_1(F+F_0)\right]$$

$$10\cos\left(\frac{5\pi n}{14}\right) = 10\cos\left(2\pi\left(\frac{5}{28}\right)n\right) \xleftrightarrow{\mathcal{F}} 5\left[\delta_1(F-5/28)+\delta_1(F+5/28)\right] = A\left[\delta_1(F-a)+\delta_1(F+a)\right]$$

$$A = 5$$
$$a = 5/28 = 0.1786$$

(b) $$4\left(\delta[n-3]-\delta[n+3]\right) \xleftrightarrow{\mathcal{F}} A\sin(aF)$$

$$4\left(\delta[n-3]-\delta[n+3]\right) \xleftrightarrow{\mathcal{F}} 4\left(e^{-j6\pi F}-e^{j6\pi F}\right) = -j8\sin(6\pi F) = A\sin(aF)$$

$$A = -j8$$
$$a = 6\pi = 18.85$$

(c) $$2\cos\left(\frac{2\pi n}{24}\right)\cos\left(\frac{2\pi n}{4}\right) \xleftrightarrow{\mathcal{F}} A\left[\delta_1(F-a)+\delta_1(F+a)+\delta_1(F-b)+\delta_1(F+b)\right]$$

$$2\cos\left(\frac{2\pi n}{24}\right)\cos\left(\frac{2\pi n}{4}\right) \xleftrightarrow{\mathcal{F}} 2\times\frac{1}{2}\left[\delta_1(F-1/24)+\delta_1(F+1/24)\right] \circledast \frac{1}{2}\left[\delta_1(F-1/4)+\delta_1(F+1/4)\right]$$

$$2\cos\left(\frac{2\pi n}{24}\right)\cos\left(\frac{2\pi n}{4}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta_1(F-1/24)+\delta_1(F+1/24)\right] * \left[\delta(F-1/4)+\delta(F+1/4)\right]$$

$$2\cos\left(\frac{2\pi n}{24}\right)\cos\left(\frac{2\pi n}{4}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\begin{bmatrix}\delta_1(F-1/4-1/24)+\delta_1(F+1/4-1/24)\\ +\delta_1(F-1/4+1/24)+\delta_1(F+1/4+1/24)\end{bmatrix}$$

$$2\cos\left(\frac{2\pi n}{24}\right)\cos\left(\frac{2\pi n}{4}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta_1(F-7/24)+\delta_1(F+5/24)+\delta_1(F-5/24)+\delta_1(F+7/24)\right]$$

$$2\cos\left(\frac{2\pi n}{24}\right)\cos\left(\frac{2\pi n}{4}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta_1(F-7/24)+\delta_1(F+7/24)+\delta_1(F-5/24)+\delta_1(F+5/24)\right]$$

$$\begin{array}{lcl} A = 1/2 = 0.5 & & A = 1/2 = 0.5 \\ a = 7/24 = 0.2917 & \text{or} & a = 5/24 = 0.2083 \\ b = 5/24 = 0.2083 & & b = 7/24 = 0.2917 \end{array}$$

Either set of answers is correct.

3. Find the numerical signal energy of $\mathrm{x}(t) = 7\,\mathrm{sinc}(3t/19)$.
(Use of Parseval's theorem is recommended, but not required.)

$$\mathrm{x}(t) = 7\,\mathrm{sinc}(3t/19) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = 7 \times (19/3)\,\mathrm{rect}(19f/3)$$

By Parseval's theorem,

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2\,dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2\,df$$

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} |7 \times (19/3)\,\mathrm{rect}(19f/3)|^2\,df = \left(\frac{7 \times 19}{3}\right)^2 \int_{-3/38}^{3/38} df = 1965.44 \times \frac{3}{19} = 310.333$$

4. Find the numerical fundamental period of $\mathrm{x}[n] = 2\cos\left(\frac{2\pi n}{24}\right) + 5\sin\left(\frac{3\pi n}{32}\right)$.

(If a single sinusoid is expressed as $\sin\left(2\pi \frac{m}{N_0} n\right)$ or $\cos\left(2\pi \frac{m}{N_0} n\right)$ where m is an integer and N_0 is an integer, and m/N_0 cannot be reduced to a simpler fraction by canceling common factors, then N_0 is its fundamental period.)

To recognize both individual fundamental periods of the two sinusoids put them in the standard form.

$$\mathrm{x}[n] = 2\cos\left(2\pi\left(\frac{1}{24}\right)n\right) + 5\sin\left(2\pi\left(\frac{3}{64}\right)n\right)$$

The two fundamental periods of the individual sinusoids are 24 and 64. The least common multiple of 24 and 64 is 192. Therefore the fundamental period of x is 192.

5. The impulse response of a discrete-time system is

$$\mathrm{h}[n] = \left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n].$$

Is it stable? Circle correct answer. Yes $\boxed{\text{No}}$

Explain your answer analytically or graphically or in a combination of these methods by examining whether or not the impulse response is absolutely summable. (A correct answer without a correct explanation gets no credit.)

Explanation:

The summation of the absolute value of $\mathrm{h}[n]$ is

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = \sum_{n=0}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\right|$$

Since the impulse response never goes negative, the magnitude of the difference is just the difference and the summation of the magnitude of the difference is the difference of the summations.

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = \sum_{n=0}^{\infty} \left(2\delta_{13}[n] - \delta_{13}[n-4]\right)$$

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = 2\sum_{n=0}^{\infty} \delta_{13}[n] - \sum_{n=0}^{\infty} \delta_{13}[n-4]$$

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = 2\left(\mathrm{u}[n] + \mathrm{u}[n-13] + \mathrm{u}[n-26] + \cdots\right) - \left(\mathrm{u}[n-4] + \mathrm{u}[n-17] + \mathrm{u}[n-30] + \cdots\right)$$

This function increases by two every 13 discrete-time units followed by a decrease by one, four discrete-time units later. It is always one larger at $n+13$ than it was at n for any $n \ge 0$. Therefore the summation is unbounded (does not converge) and the system is unstable.

6. A continuous-time system is described by the differential equation

$$0.4\,\mathrm{y}'(t) + 0.8\,\mathrm{y}(t) = 2\,\mathrm{x}'(t) - 7\,\mathrm{x}(t)$$

where x is the excitation and y is the response. The impulse response can be written in the form

$$\mathrm{h}(t) = K_{\mathrm{h}} e^{-t/\tau}\,\mathrm{u}(t) + K_{\delta}\delta(t).$$

Find the numerical values of the constants K_{h}, K_{δ} and τ. (If you have time it would be good to substitute the solution back into the differential equation and check to see that the two sides are equal. But this is not required.)

Given the form of h, $\tau = -1/\lambda$ where λ is the eigenvalue. The eigenvalue is the solution of $0.4\lambda + 0.8 = 0 \Rightarrow \lambda = -2$. Therefore $\tau = 1/2$.

$$0.4\,\mathrm{h}'(t) + 0.8\,\mathrm{h}(t) = 2\delta'(t) - 7\delta(t)$$

$$0.4\left[\underbrace{\mathrm{h}(0^+)}_{K_{\mathrm{h}}} - \underbrace{\mathrm{h}(0^-)}_{=0}\right] + 0.8\underbrace{\int_{0^-}^{0^+}\mathrm{h}(t)\,dt}_{=K_{\delta}} = 2\left[\underbrace{\delta(0^+)}_{=0} - \underbrace{\delta(0^-)}_{=0}\right] - 7\left[\underbrace{\mathrm{u}(0^+)}_{=1} - \underbrace{\mathrm{u}(0^-)}_{=0}\right]$$

$$0.4K_{\mathrm{h}} + 0.8K_{\delta} = -7$$

$$0.4\underbrace{\int_{0^-}^{0^+}\mathrm{h}(t)\,dt}_{=K_{\delta}} + 0.8\underbrace{\int_{0^-}^{0^+}\int_{-\infty}^{t}\mathrm{h}(\lambda)\,d\lambda\,dt}_{=0} = 2\left[\underbrace{\mathrm{u}(0^+)}_{=1} - \underbrace{\mathrm{u}(0^-)}_{=0}\right] - 7\left[\underbrace{\mathrm{ramp}(0^+)}_{=0} - \underbrace{\mathrm{ramp}(0^-)}_{=0}\right]$$

$$0.4K_{\delta} = 2 \Rightarrow K_{\delta} = 5 \Rightarrow 0.4K_{\mathrm{h}} + 4 = -7 \Rightarrow K_{\mathrm{h}} = -11/0.4 = -27.5$$

$$\mathrm{h}(t) = -27.5e^{-2t}\,\mathrm{u}(t) + 5\delta(t)$$

Substitute the solution into the differential equation as check whether both sides are the same.

$$\mathrm{h}'(t) = -27.5\left[e^{-2t}\delta(t) - 2e^{-2t}\,\mathrm{u}(t)\right] + 5\delta'(t)$$

$$0.4\left\{-27.5\left[e^{-2t}\delta(t) - 2e^{-2t}\,\mathrm{u}(t)\right] + 5\delta'(t)\right\} + 0.8\left[-27.5e^{-2t}\,\mathrm{u}(t) + 5\delta(t)\right] = 2\delta'(t) - 7\delta(t)$$

$$-11\delta(t) + 22e^{-2t}\,\mathrm{u}(t) + 2\delta'(t) - 22e^{-2t}\,\mathrm{u}(t) + 4\delta(t) = 2\delta'(t) - 7\delta(t)$$

$$2\delta'(t) - 7\delta(t) = 2\delta'(t) - 7\delta(t)$$ Check. Correct solution.

Solution of ECE 315 Final Examination Su08

1. Find the numerical values of the constants,

(a) $$A\sin(300\pi t+\pi/3)\xleftrightarrow{\mathcal{F}} j6\left[\delta(f+a)-\delta(f-a)\right]e^{bf}$$

$$\sin(2\pi f_0 t)\xleftrightarrow{\mathcal{F}}\frac{j}{2}\left[\delta(f+f_0)-\delta(f-f_0)\right]$$

Multiply both sides by 12. $12\sin(2\pi f_0 t)\xleftrightarrow{\mathcal{F}} j6\left[\delta(f+f_0)-\delta(f-f_0)\right]$

Identify $f_0=150$. $12\sin(300\pi t)\xleftrightarrow{\mathcal{F}} j6\left[\delta(f+150)-\delta(f-150)\right]$

Use the time-shifting property.

$$12\sin(200\pi(t-t_0))\xleftrightarrow{\mathcal{F}} j6\left[\delta(f+150)-\delta(f-150)\right]e^{-j2\pi f t_0}$$

$$A=12\ ,\ a=150$$

Solve for t_0.

$$12\sin(300\pi t-300\pi t_0)\xleftrightarrow{\mathcal{F}} j6\left[\delta(f+150)-\delta(f-150)\right]e^{-j2\pi f t_0}$$

$$-300\pi t_0=\pi/3$$

$$t_0=-\frac{\pi/3}{300\pi}=-\frac{1}{900}\Rightarrow b=-j2\pi t_0=(-j)\left(-\frac{2\pi}{900}\right)=j\frac{\pi}{450}\ \text{ or }\ j0.00698$$

(b) $$A\,\text{rect}(t/a)*\text{rect}(t/b)\xleftrightarrow{\mathcal{F}}30\,\text{sinc}(3f)\text{sinc}(5f)$$

$$A\,\text{rect}(t/a)*\text{rect}(t/b)\xleftrightarrow{\mathcal{F}}A\times a\,\text{sinc}(af)\times b\,\text{sinc}(bf)=30\,\text{sinc}(3f)\text{sinc}(5f)$$

$$\begin{aligned}&A\times a\times b=30\\&a=3\\&b=5\Rightarrow A=30/15=2\end{aligned}\qquad\text{or}\qquad\begin{aligned}&A\times a\times b=30\\&a=5\\&b=3\Rightarrow A=30/15=2\end{aligned}$$

Either set of answers is correct.

(c) $$\frac{d}{dt}(7\,\text{tri}(3t))\xleftrightarrow{\mathcal{F}}Af\,\text{sinc}^2(af)$$

$$\frac{d}{dt}(7\,\text{tri}(3t))\xleftrightarrow{\mathcal{F}}j2\pi f\times(7/3)\text{sinc}^2(f/3)=Af\,\text{sinc}^2(af)$$

$$\begin{aligned}&A=j2\pi\times(7/3)=j14\pi/3=j14.66\\&a=1/3\end{aligned}$$

2. Find the numerical values of the constants.

(a) $8\cos\left(\frac{9\pi n}{14}\right) \xleftrightarrow{\mathcal{F}} A\left[\delta_1(F-a)+\delta_1(F+a)\right]$

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta_1(F-F_0)+\delta_1(F+F_0)\right]$$

$$8\cos\left(\frac{9\pi n}{14}\right) = 8\cos\left(2\pi\left(\frac{9}{28}\right)n\right) \xleftrightarrow{\mathcal{F}} 4\left[\delta_1(F-9/28)+\delta_1(F+9/28)\right] = A\left[\delta_1(F-a)+\delta_1(F+a)\right]$$

$$A = 4$$
$$a = 9/28 = 0.3214$$

(b) $12\left(\delta[n-4]-\delta[n+4]\right) \xleftrightarrow{\mathcal{F}} A\sin(aF)$

$$12\left(\delta[n-4]-\delta[n+4]\right) \xleftrightarrow{\mathcal{F}} 12\left(e^{-j8\pi F}-e^{j8\pi F}\right) = -j24\sin(8\pi F) = A\sin(aF)$$

$$A = -j24$$
$$a = 8\pi = 25.133$$

(c) $5\cos\left(\frac{2\pi n}{18}\right)\cos\left(\frac{2\pi n}{6}\right) \xleftrightarrow{\mathcal{F}} A\left[\delta_1(F-a)+\delta_1(F+a)+\delta_1(F-b)+\delta_1(F+b)\right]$

$$5\cos\left(\frac{2\pi n}{18}\right)\cos\left(\frac{2\pi n}{6}\right) \xleftrightarrow{\mathcal{F}} 5\times\frac{1}{2}\left[\delta_1(F-1/18)+\delta_1(F+1/18)\right] \circledast \frac{1}{2}\left[\delta_1(F-1/6)+\delta_1(F+1/6)\right]$$

$$5\cos\left(\frac{2\pi n}{18}\right)\cos\left(\frac{2\pi n}{6}\right) \xleftrightarrow{\mathcal{F}} \frac{5}{4}\left[\delta_1(F-1/18)+\delta_1(F+1/18)\right] * \left[\delta(F-1/6)+\delta(F+1/6)\right]$$

$$5\cos\left(\frac{2\pi n}{18}\right)\cos\left(\frac{2\pi n}{6}\right) \xleftrightarrow{\mathcal{F}} \frac{5}{4}\left[\begin{array}{l}\delta_1(F-1/6-1/18)+\delta_1(F+1/6-1/18)\\ +\delta_1(F-1/6+1/18)+\delta_1(F+1/6+1/18)\end{array}\right]$$

$$5\cos\left(\frac{2\pi n}{18}\right)\cos\left(\frac{2\pi n}{6}\right) \xleftrightarrow{\mathcal{F}} \frac{5}{4}\left[\delta_1(F-2/9)+\delta_1(F+1/9)+\delta_1(F-1/9)+\delta_1(F+2/9)\right]$$

$$5\cos\left(\frac{2\pi n}{18}\right)\cos\left(\frac{2\pi n}{6}\right) \xleftrightarrow{\mathcal{F}} \frac{5}{4}\left[\delta_1(F-2/9)+\delta_1(F+2/9)+\delta_1(F-1/9)+\delta_1(F+1/9)\right]$$

$$\begin{array}{lcl} A = 5/4 = 1.25 & & A = 5/4 = 1.25 \\ a = 2/9 = 0.2222 & \text{or} & a = 1/9 = 0.1111 \\ b = 1/9 = 0.1111 & & b = 2/9 = 0.2222 \end{array}$$

Either set of answers is correct.

3. Find the numerical signal energy of $\mathrm{x}(t) = 11\,\mathrm{sinc}(5t/13)$.

(Use of Parseval's theorem is recommended, but not required.)

$$\mathrm{x}(t) = 11\,\mathrm{sinc}(5t/13) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = 11 \times (13/5)\,\mathrm{rect}(13f/5)$$

By Parseval's theorem,

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} \left|\mathrm{x}(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|\mathrm{X}(f)\right|^2 df$$

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} \left|11 \times (13/5)\,\mathrm{rect}(13f/5)\right|^2 df = \left(\frac{11 \times 13}{5}\right)^2 \int_{-5/26}^{5/26} df = 817.96 \times \frac{5}{13} = 314.6$$

4. Find the numerical fundamental period of $\mathrm{x}[n] = 6\cos\left(\frac{2\pi n}{30}\right) + 5\sin\left(\frac{3\pi n}{36}\right)$.

(If a single sinusoid is expressed as $\sin\left(2\pi \frac{m}{N_0} n\right)$ or $\cos\left(2\pi \frac{m}{N_0} n\right)$ where *m* is an integer and N_0 is an integer, and m/N_0 cannot be reduced to a simpler fraction by canceling common factors, then N_0 is its fundamental period.)

To recognize both individual fundamental periods of the two sinusoids put them in the standard form.

$$\mathrm{x}[n] = 6\cos\left(2\pi\left(\frac{1}{30}\right)n\right) + 5\sin\left(2\pi\left(\frac{3}{72}\right)n\right)$$

The two fundamental periods of the individual sinusoids are 30 and 72. The least common multiple of 30 and 72 is 360. Therefore the fundamental period of x is 360.

5. The impulse response of a discrete-time system is

$$\mathrm{h}[n] = \left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n].$$

Is it stable? Circle correct answer. Yes $\boxed{\text{No}}$

Explain your answer analytically or graphically or in a combination of these methods by examining whether or not the impulse response is absolutely summable. (A correct answer without a correct explanation gets no credit.)

Explanation:

The summation of the absolute value of $\mathrm{h}[n]$ is

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = \sum_{n=0}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\right|$$

Since the impulse response never goes negative, the magnitude of the difference is just the difference and the summation of the magnitude of the difference is the difference of the summations.

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = \sum_{n=0}^{\infty} \left(2\delta_{13}[n] - \delta_{13}[n-4]\right)$$

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = 2\sum_{n=0}^{\infty} \delta_{13}[n] - \sum_{n=0}^{\infty} \delta_{13}[n-4]$$

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = 2\left(\mathrm{u}[n] + \mathrm{u}[n-13] + \mathrm{u}[n-26] + \cdots\right) - \left(\mathrm{u}[n-4] + \mathrm{u}[n-17] + \mathrm{u}[n-30] + \cdots\right)$$

This function increases by two every 13 discrete-time units followed by a decrease by one, four discrete-time units later. It is always one larger at $n+13$ than it was at n for any $n \geq 0$. Therefore the summation is unbounded (does not converge) and the system is unstable.

6. A continuous-time system is described by the differential equation

$$0.6\,y'(t) + 0.9\,y(t) = 2\,x'(t) - 3\,x(t)$$

where x is the excitation and y is the response. The impulse response can be written in the form

$$h(t) = K_h e^{-t/\tau}\,u(t) + K_\delta \delta(t).$$

Find the numerical values of the constants K_h, K_δ and τ. (If you have time it would be good to substitute the solution back into the differential equation and check to see that the two sides are equal. But this is not required.)

Given the form of h, $\tau = -1/\lambda$ where λ is the eigenvalue. The eigenvalue is the solution of $0.6\lambda + 0.9 = 0 \Rightarrow \lambda = -3/2$. Therefore $\tau = 2/3$.

$$0.6\,h'(t) + 0.9\,h(t) = 2\delta'(t) - 3\delta(t)$$

$$0.6\left[\underbrace{h(0^+)}_{K_h} - \underbrace{h(0^-)}_{=0}\right] + 0.9\underbrace{\int_{0^-}^{0^+} h(t)\,dt}_{=K_\delta} = 2\left[\underbrace{\delta(0^+)}_{=0} - \underbrace{\delta(0^-)}_{=0}\right] - 3\left[\underbrace{u(0^+)}_{=1} - \underbrace{u(0^-)}_{=0}\right]$$

$$0.6K_h + 0.9K_\delta = -3$$

$$0.6\underbrace{\int_{0^-}^{0^+} h(t)\,dt}_{=K_\delta} + 0.9\underbrace{\int_{0^-}^{0^+}\int_{-\infty}^{t} h(\lambda)\,d\lambda\,dt}_{=0} = 2\left[\underbrace{u(0^+)}_{=1} - \underbrace{u(0^-)}_{=0}\right] - 3\left[\underbrace{\mathrm{ramp}(0^+)}_{=0} - \underbrace{\mathrm{ramp}(0^-)}_{=0}\right]$$

$$0.6K_\delta = 2 \Rightarrow K_\delta = 10/3 \Rightarrow 0.6K_h + 3 = -3 \Rightarrow K_h = -6/0.6 = -10$$

$$h(t) = -10e^{-3t/2}\,u(t) + (10/3)\delta(t)$$

Substitute the solution into the differential equation as check whether both sides are the same.

$$h'(t) = -10\left[e^{-3t/2}\delta(t) - (3/2)e^{-3t/2}\,u(t)\right] + (10/3)\delta'(t)$$

$$0.6\left\{-10\left[e^{-3t/2}\delta(t) - (3/2)e^{-3t/2}\,u(t)\right] + (10/3)\delta'(t)\right\} + 0.9\left[-10e^{-3t/2}\,u(t) + (10/3)\delta(t)\right] = 2\delta'(t) - 3\delta(t)$$

$$-6\delta(t) + 9e^{-3t/2}\,u(t) + 2\delta'(t) - 9e^{-3t/2}\,u(t) + 3\delta(t) = 2\delta'(t) - 3\delta(t)$$

$$2\delta'(t) - 3\delta(t) = 2\delta'(t) - 3\delta(t)$$ Check. Correct solution.

Solution of ECE 315 Final Examination Su08

1. Find the numerical values of the constants,

(a) $A\sin(100\pi t + \pi/3) \xleftrightarrow{\mathcal{F}} j4[\delta(f+a) - \delta(f-a)]e^{bf}$

$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}[\delta(f+f_0) - \delta(f-f_0)]$$

Multiply both sides by 8. $8\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} j4[\delta(f+f_0) - \delta(f-f_0)]$

Identify $f_0 = 50$. $8\sin(100\pi t) \xleftrightarrow{\mathcal{F}} j4[\delta(f+50) - \delta(f-50)]$

Use the time-shifting property.

$$8\sin(100\pi(t-t_0)) \xleftrightarrow{\mathcal{F}} j4[\delta(f+50) - \delta(f-50)]e^{-j2\pi f t_0}$$

$$A = 8 \ , \ a = 50$$

Solve for t_0.

$$8\sin(100\pi t - 100\pi t_0) \xleftrightarrow{\mathcal{F}} j4[\delta(f+50) - \delta(f-50)]e^{-j2\pi f t_0}$$

$$-100\pi t_0 = \pi/3$$

$$t_0 = -\frac{\pi/3}{100\pi} = -\frac{1}{300} \Rightarrow b = -j2\pi t_0 = (-j)\left(-\frac{2\pi}{300}\right) = j\frac{\pi}{150} \ \text{ or } \ j0.02094$$

(b) $A\,\text{rect}(t/a) * \text{rect}(t/b) \xleftrightarrow{\mathcal{F}} 600\,\text{sinc}(8f)\,\text{sinc}(10f)$

$$A\,\text{rect}(t/a) * \text{rect}(t/b) \xleftrightarrow{\mathcal{F}} A \times a\,\text{sinc}(af) \times b\,\text{sinc}(bf) = 600\,\text{sinc}(8f)\,\text{sinc}(10f)$$

$$\begin{array}{lcl} A \times a \times b = 600 & & A \times a \times b = 600 \\ a = 8 & \text{or} & a = 10 \\ b = 10 \Rightarrow A = 600/80 = 7.5 & & b = 8 \Rightarrow A = 600/80 = 7.5 \end{array}$$

Either set of answers is correct.

(c) $\frac{d}{dt}(24\,\text{tri}(9t)) \xleftrightarrow{\mathcal{F}} Af\,\text{sinc}^2(af)$

$$\frac{d}{dt}(24\,\text{tri}(9t)) \xleftrightarrow{\mathcal{F}} j2\pi f \times (24/9)\,\text{sinc}^2(f/9) = Af\,\text{sinc}^2(af)$$

$$A = j2\pi \times (24/9) = j48\pi/9 = j16.755$$
$$a = 1/9$$

2. Find the numerical values of the constants.

(a) $5\cos\left(\frac{5\pi n}{18}\right) \xleftrightarrow{\mathcal{F}} A\left[\delta_1(F-a)+\delta_1(F+a)\right]$

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta_1(F-F_0)+\delta_1(F+F_0)\right]$$

$$5\cos\left(\frac{5\pi n}{18}\right) = 5\cos\left(2\pi\left(\frac{5}{36}\right)n\right) \xleftrightarrow{\mathcal{F}} \frac{5}{2}\left[\delta_1(F-5/36)+\delta_1(F+5/36)\right] = A\left[\delta_1(F-a)+\delta_1(F+a)\right]$$

$$A = 5/2 = 2.5$$
$$a = 5/36 = 0.1389$$

(b) $22\left(\delta[n-7]-\delta[n+7]\right) \xleftrightarrow{\mathcal{F}} A\sin(aF)$

$$22\left(\delta[n-7]-\delta[n+7]\right) \xleftrightarrow{\mathcal{F}} 22\left(e^{-j14\pi F}-e^{j14\pi F}\right) = -j44\sin(14\pi F) = A\sin(aF)$$

$$A = -j44$$
$$a = 14\pi = 43.98$$

(c) $9\cos\left(\frac{2\pi n}{36}\right)\cos\left(\frac{2\pi n}{9}\right) \xleftrightarrow{\mathcal{F}} A\left[\delta_1(F-a)+\delta_1(F+a)+\delta_1(F-b)+\delta_1(F+b)\right]$

$$9\cos\left(\frac{2\pi n}{36}\right)\cos\left(\frac{2\pi n}{9}\right) \xleftrightarrow{\mathcal{F}} 9\times\frac{1}{2}\left[\delta_1(F-1/36)+\delta_1(F+1/36)\right] \circledast \frac{1}{2}\left[\delta_1(F-1/9)+\delta_1(F+1/9)\right]$$

$$9\cos\left(\frac{2\pi n}{36}\right)\cos\left(\frac{2\pi n}{9}\right) \xleftrightarrow{\mathcal{F}} \frac{9}{4}\left[\delta_1(F-1/36)+\delta_1(F+1/36)\right] * \left[\delta(F-1/9)+\delta(F+1/9)\right]$$

$$9\cos\left(\frac{2\pi n}{36}\right)\cos\left(\frac{2\pi n}{9}\right) \xleftrightarrow{\mathcal{F}} \frac{9}{4}\begin{bmatrix}\delta_1(F-1/9-1/36)+\delta_1(F+1/9-1/36)\\ +\delta_1(F-1/9+1/36)+\delta_1(F+1/9+1/36)\end{bmatrix}$$

$$9\cos\left(\frac{2\pi n}{36}\right)\cos\left(\frac{2\pi n}{9}\right) \xleftrightarrow{\mathcal{F}} \frac{9}{4}\left[\delta_1(F-5/36)+\delta_1(F+1/12)+\delta_1(F-1/12)+\delta_1(F+5/36)\right]$$

$$9\cos\left(\frac{2\pi n}{36}\right)\cos\left(\frac{2\pi n}{9}\right) \xleftrightarrow{\mathcal{F}} \frac{9}{4}\left[\delta_1(F-5/36)+\delta_1(F+5/36)+\delta_1(F-1/12)+\delta_1(F+1/12)\right]$$

$A = 9/4 = 2.25$, $a = 5/36 = 0.1389$, $b = 1/12 = 0.0833$ or $A = 9/4 = 2.25$, $a = 1/12 = 0.0833$, $b = 5/36 = 0.1389$

Either set of answers is correct.

3. Find the numerical signal energy of $\mathrm{x}(t) = 8\,\mathrm{sinc}(4t/11)$.
(Use of Parseval's theorem is recommended, but not required.)

$$\mathrm{x}(t) = 8\,\mathrm{sinc}(4t/11) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = 8 \times (11/4)\,\mathrm{rect}(11f/4)$$

By Parseval's theorem,

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2\,dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2\,df$$

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} |8 \times (11/4)\,\mathrm{rect}(11f/4)|^2\,df = \left(\frac{8 \times 11}{4}\right)^2 \int_{-2/11}^{2/11} df = 484 \times \frac{4}{11} = 176$$

4. Find the numerical fundamental period of $\mathrm{x}[n] = 2\cos\left(\frac{2\pi n}{36}\right) - 12\sin\left(\frac{3\pi n}{64}\right)$.

(If a single sinusoid is expressed as $\sin\left(2\pi \frac{m}{N_0} n\right)$ or $\cos\left(2\pi \frac{m}{N_0} n\right)$ where m is an integer and N_0 is an integer, and m/N_0 cannot be reduced to a simpler fraction by canceling common factors, then N_0 is its fundamental period.)

To recognize both individual fundamental periods of the two sinusoids put them in the standard form.

$$\mathrm{x}[n] = 2\cos\left(2\pi\left(\frac{1}{48}\right)n\right) + 5\sin\left(2\pi\left(\frac{3}{128}\right)n\right)$$

The two fundamental periods of the individual sinusoids are 48 and 128. The least common multiple of 48 and 128 is 384. Therefore the fundamental period of x is 384.

5. The impulse response of a discrete-time system is

$$\mathrm{h}[n] = \left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n].$$

Is it stable? Circle correct answer. Yes $\boxed{\text{No}}$

Explain your answer analytically or graphically or in a combination of these methods by examining whether or not the impulse response is absolutely summable. (A correct answer without a correct explanation gets no credit.)

Explanation:

The summation of the absolute value of $\mathrm{h}[n]$ is

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = \sum_{n=0}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\right|$$

Since the impulse response never goes negative, the magnitude of the difference is just the difference and the summation of the magnitude of the difference is the difference of the summations.

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = \sum_{n=0}^{\infty} \left(2\delta_{13}[n] - \delta_{13}[n-4]\right)$$

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = 2\sum_{n=0}^{\infty} \delta_{13}[n] - \sum_{n=0}^{\infty} \delta_{13}[n-4]$$

$$\sum_{n=-\infty}^{\infty} \left|\left(2\delta_{13}[n] - \delta_{13}[n-4]\right)\mathrm{u}[n]\right| = 2\left(\mathrm{u}[n] + \mathrm{u}[n-13] + \mathrm{u}[n-26] + \cdots\right) - \left(\mathrm{u}[n-4] + \mathrm{u}[n-17] + \mathrm{u}[n-30] + \cdots\right)$$

This function increases by two every 13 discrete-time units followed by a decrease by one, four discrete-time units later. It is always one larger at $n+13$ than it was at n for any $n \geq 0$. Therefore the summation is unbounded (does not converge) and the system is unstable.

6. A continuous-time system is described by the differential equation

$$0.1\mathrm{y}'(t)+0.4\,\mathrm{y}(t)=5\mathrm{x}'(t)-4\,\mathrm{x}(t)$$

where x is the excitation and y is the response. The impulse response can be written in the form

$$\mathrm{h}(t)=K_{\mathrm{h}}e^{-t/\tau}\,\mathrm{u}(t)+K_{\delta}\delta(t).$$

Find the numerical values of the constants K_{h}, K_{δ} and τ. (If you have time it would be good to substitute the solution back into the differential equation and check to see that the two sides are equal. But this is not required.)

Given the form of h, $\tau=-1/\lambda$ where λ is the eigenvalue. The eigenvalue is the solution of $0.1\lambda+0.4=0 \Rightarrow \lambda=-4$. Therefore $\tau=1/4$.

$$0.1\mathrm{h}'(t)+0.4\,\mathrm{h}(t)=5\delta'(t)-4\delta(t)$$

$$0.1\Big[\underbrace{\mathrm{h}(0^+)}_{K_{\mathrm{h}}}-\underbrace{\mathrm{h}(0^-)}_{=0}\Big]+0.4\underbrace{\int_{0^-}^{0^+}\mathrm{h}(t)\,dt}_{=K_\delta}=5\Big[\underbrace{\delta(0^+)}_{=0}-\underbrace{\delta(0^-)}_{=0}\Big]-4\Big[\underbrace{\mathrm{u}(0^+)}_{=1}-\underbrace{\mathrm{u}(0^-)}_{=0}\Big]$$

$$0.1K_{\mathrm{h}}+0.4K_{\delta}=-4$$

$$0.1\underbrace{\int_{0^-}^{0^+}\mathrm{h}(t)\,dt}_{=K_\delta}+0.4\underbrace{\int_{0^-}^{0^+}\int_{-\infty}^{t}\mathrm{h}(\lambda)\,d\lambda\,dt}_{=0}=5\Big[\underbrace{\mathrm{u}(0^+)}_{=1}-\underbrace{\mathrm{u}(0^-)}_{=0}\Big]-4\Big[\underbrace{\mathrm{ramp}(0^+)}_{=0}-\underbrace{\mathrm{ramp}(0^-)}_{=0}\Big]$$

$$0.1K_{\delta}=5 \Rightarrow K_{\delta}=50 \Rightarrow 0.1K_{\mathrm{h}}+20=-4 \Rightarrow K_{\mathrm{h}}=-24/0.1=-240$$

$$\mathrm{h}(t)=-240e^{-4t}\,\mathrm{u}(t)+50\delta(t)$$

Substitute the solution into the differential equation as check whether both sides are the same.

$$\mathrm{h}'(t)=-240\left[e^{-4t}\delta(t)-4e^{-4t}\,\mathrm{u}(t)\right]+50\delta'(t)$$

$$0.1\left\{-240\left[e^{-4t}\delta(t)-4e^{-4t}\,\mathrm{u}(t)\right]+50\delta'(t)\right\}+0.4\left[-240e^{-4t}\,\mathrm{u}(t)+50\delta(t)\right]=5\delta'(t)-4\delta(t)$$

$$-4\delta(t)+5\delta'(t)=5\delta'(t)-4\delta(t)$$ Check. Correct solution.