Solution of ECE 315 Final Examination Su10

1. Find the numerical values of the constants.

(a) $8(\mathrm{u}[n+3]-\mathrm{u}[n-2])*\delta_{12}[n] \xleftrightarrow[12]{\mathcal{DFT}} Ae^{bk}\,\mathrm{drcl}(ck,D)$ (k is harmonic number)

Using $(\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1])*\delta_N[n] \xleftrightarrow[mN]{\mathcal{DFT}} \dfrac{e^{-j\pi k(n_1+n_0)/mN}}{e^{-j\pi k/mN}}(n_1-n_0)m\,\mathrm{drcl}(k/mN, n_1-n_0)\delta_m[k]$

with $m=1$, $n_0=-3$, $n_1=2$, $N=12$

$(\mathrm{u}[n+3]-\mathrm{u}[n-2])*\delta_{12}[n] \xleftrightarrow[12]{\mathcal{DFT}} 5e^{j\pi k/6}\,\mathrm{drcl}(k/12,5)$

$8(\mathrm{u}[n+3]-\mathrm{u}[n-2])*\delta_{12}[n] \xleftrightarrow[12]{\mathcal{DFT}} 40e^{j\pi k/6}\,\mathrm{drcl}(k/12,5)$

$A=40$, $b=j\pi/6$, $c=1/12$, $D=5$

(b) $8(\mathrm{u}[n+3]-\mathrm{u}[n-2]) \xleftrightarrow{\mathcal{F}} Ae^{b\Omega}\,\mathrm{drcl}(c\Omega,D)$

Using $\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \dfrac{e^{-j\pi F(n_0+n_1)}}{e^{-j\pi F}}(n_1-n_0)\mathrm{drcl}(F,n_1-n_0)$ with $n_0=-3$, $n_1=2$ and $\Omega=2\pi F$

$\mathrm{u}[n+3]-\mathrm{u}[n-2] \xleftrightarrow{\mathcal{F}} 5e^{j\Omega}\,\mathrm{drcl}(\Omega/2\pi,5)$

$8(\mathrm{u}[n+3]-\mathrm{u}[n-2]) \xleftrightarrow{\mathcal{F}} 40e^{j\Omega}\,\mathrm{drcl}(\Omega/2\pi,5)$

$A=40$, $b=j$, $c=1/2\pi$, $D=5$

2. Let $x[n] \xleftrightarrow{\mathcal{F}} X(F) = 8\,\mathrm{tri}(2F)e^{-j2\pi F} * \delta_1(F)$, a phase-shifted triangle in the range $-1/2 < F < 1/2$ that repeats that pattern periodically, with fundamental period one. Also let

$$y[n] = \begin{cases} x[n/3] & , \text{ if } n/3 \text{ is an integer} \\ 0 & , \text{ if } n/3 \text{ is not an integer} \end{cases}$$

and let $y[n] \xleftrightarrow{\mathcal{F}} Y(F)$.

(a) Find the numerical magnitude and angle (in radians) of $X(0.3)$.

$X(0.3) = 8\,\mathrm{tri}(0.6)e^{-j2\pi(0.3)} = 3.2\angle -1.885$

(b) Find the numerical magnitude and angle (in radians) of $X(2.2)$.

Taking advantage of the periodicity of $X(F)$

$$X(2.2) = X(2.2-2) = X(0.2) = 8\,\mathrm{tri}(0.4)e^{-j2\pi(0.2)} = 4.8\angle -1.2566 .$$

(c) What is the numerical fundamental period of $Y(F)$?

Use the property

$$\text{If } z[n] = \left\{ \begin{matrix} x[n/m] & , & n/m \text{ an integer} \\ 0 & , & \text{otherwise} \end{matrix} \right\} \ , \ z[n] \xleftrightarrow{\mathcal{F}} X(mF)$$

$Y(F)$ is a compressed version of $X(F)$ and the compression factor is 3. So the value of $X(F)$ at $F = 1/2$ is the same as the value of $Y(F)$ at $F = 1/6$ and the fundamental period of $Y(F)$ is therefore 1/3.

Alternate Solution: $Y(F) = X(3F) = 8\,\mathrm{tri}(6F)e^{-j6\pi F} * \delta_1(3F)$

Using the scaling property of the periodic impulse and the scaling property of convolution

$Y(F) = 3 \times 8\,\mathrm{tri}(6F)e^{-j6\pi F} * (1/3)\delta_{1/3}(F) = 8\,\mathrm{tri}(6F)e^{-j6\pi F} * \delta_{1/3}(F)$

(d) Find the numerical magnitude and angle (in radians) of $Y(0.55)$.

Using the fact that the fundamental period is 1/3,

$Y(0.55) = Y(0.55 - 2/3) = Y(-0.1167) = 8\,\mathrm{tri}(-6 \times 0.1167)e^{j6\pi \times 0.1167} = 2.3984\angle 2.1997$

(e) Using Parseval's theorem $\sum_{n=-\infty}^{\infty}\left|\mathrm{x}[n]\right|^2 = \int_1 \left|\mathrm{X}(F)\right|^2 dF$ find the numerical signal energies of both $\mathrm{x}[n]$ and $\mathrm{y}[n]$.

$$E_{\mathrm{x}} = \int_1 \left|\mathrm{X}(F)\right|^2 dF = \int_{-1/2}^{1/2} \left|8\,\mathrm{tri}(2F)e^{-j2\pi F}\right|^2 dF = 64\int_{-1/2}^{1/2} \mathrm{tri}^2(2F)dF = 128\int_{0}^{1/2}(1-2F)^2\,dF$$

$$E_{\mathrm{x}} = 128\int_{0}^{1/2}\left(1-4F+4F^2\right)dF = 128\left[F-2F^2+4F^3/3\right]_0^{1/2} = 128\left[1/2-1/2+1/6\right] = 128/6 = 21.333\ldots$$

$$E_{\mathrm{y}} = \int_1 \left|\mathrm{Y}(F)\right|^2 dF = \int_{-1/2}^{1/2} \left|8\,\mathrm{tri}(6F)e^{-j6\pi F} * \delta_{1/3}(F)\right|^2 dF = 3\int_{-1/6}^{1/6} \left|8\,\mathrm{tri}(6F)e^{-j6\pi F}\right|^2 dF = 192\int_{-1/6}^{1/6}\mathrm{tri}^2(6F)dF = 384\int_{0}^{1/6}(1-6F)^2\,dF$$

$$E_{\mathrm{x}} = 384\int_{0}^{1/6}\left(1-12F+36F^2\right)dF = 384\left[F-6F^2+12F^3\right]_0^{1/6} = 384\left[1/6-1/6+12/216\right] = 21.333\ldots$$

3. A periodic discrete-time signal is described over two fundamental periods by

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$\mathrm{x}[n]$	$\cdots$	a	b	c	d	a	b	c	d	$\cdots$

The DFT of $\mathrm{x}[n]$ is $\mathrm{X}[k] = \sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N}$. Fill in the table for $\mathrm{X}[k]$ below with expressions for the real and imaginary parts of $\mathrm{X}[k]$ written as linear combinations of a, b, c and d.

$$\mathrm{X}[k] = \sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N} = \sum_{n=0}^{2}\mathrm{x}[n]e^{-j2\pi kn/3}$$

$$\mathrm{X}[0] = a+b+c+d+j0$$
$$\mathrm{X}[1] = c-jd-a+jb = c-a+j(b-d)$$
$$\mathrm{X}[2] = c-d+a-b+j0$$
$$\mathrm{X}[3] = c-a+j(d-b)$$

Solution of ECE 315 Final Examination Su10

1. Find the numerical values of the constants.

(a) $8\left(\mathrm{u}[n+3]-\mathrm{u}[n-1]\right)*\delta_{10}[n] \xleftrightarrow[10]{\mathcal{DFT}} Ae^{bk}\,\mathrm{drcl}(ck,D)$ (k is harmonic number)

Using $\left(\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1]\right)*\delta_N[n] \xleftrightarrow[mN]{\mathcal{DFT}} \dfrac{e^{-j\pi k(n_1+n_0)/mN}}{e^{-j\pi k/mN}}(n_1-n_0)m\,\mathrm{drcl}(k/mN,n_1-n_0)\delta_m[k]$

with $m=1$, $n_0=-3$, $n_1=1$, $N=10$

$\left(\mathrm{u}[n+3]-\mathrm{u}[n-1]\right)*\delta_{10}[n] \xleftrightarrow[10]{\mathcal{DFT}} 4e^{j3\pi k/10}\,\mathrm{drcl}(k/10,4)$

$8\left(\mathrm{u}[n+3]-\mathrm{u}[n-1]\right)*\delta_{10}[n] \xleftrightarrow[10]{\mathcal{DFT}} 32e^{j3\pi k/10}\,\mathrm{drcl}(k/10,4)$

$A=32$, $b=j3\pi/10$, $c=1/10$, $D=4$

(b) $8\left(\mathrm{u}[n+3]-\mathrm{u}[n-1]\right) \xleftrightarrow{\mathcal{F}} Ae^{b\Omega}\,\mathrm{drcl}(c\Omega,D)$

Using $\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \dfrac{e^{-j\pi F(n_0+n_1)}}{e^{-j\pi F}}(n_1-n_0)\mathrm{drcl}(F,n_1-n_0)$ with $n_0=-3$, $n_1=1$ and $\Omega=2\pi F$

$\mathrm{u}[n+3]-\mathrm{u}[n-2] \xleftrightarrow{\mathcal{F}} 4e^{j3\Omega/2}\,\mathrm{drcl}(\Omega/2\pi,4)$

$8\left(\mathrm{u}[n+3]-\mathrm{u}[n-2]\right) \xleftrightarrow{\mathcal{F}} 32e^{j3\Omega/2}\,\mathrm{drcl}(\Omega/2\pi,4)$

$A=32$, $b=j3/2$, $c=1/2\pi$, $D=4$

2. Let $\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F) = 10\,\mathrm{tri}(2F)e^{-j4\pi F} * \delta_1(F)$, a phase-shifted triangle in the range $-1/2 < F < 1/2$ that repeats that pattern periodically, with fundamental period one. Also let

$$\mathrm{y}[n] = \begin{cases} \mathrm{x}[n/4] & \text{, if } n/4 \text{ is an integer} \\ 0 & \text{, if } n/4 \text{ is not an integer} \end{cases}$$

and let $\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{Y}(F)$.

(a) Find the numerical magnitude and angle (in radians) of $\mathrm{X}(0.3)$.

$\mathrm{X}(0.3) = 10\,\mathrm{tri}(0.6)e^{-j4\pi(0.3)} = 4\angle 2.5133$ or $4\angle\text{-}3.77$

(b) Find the numerical magnitude and angle (in radians) of $\mathrm{X}(2.2)$.

Taking advantage of the periodicity of $\mathrm{X}(F)$

$$\mathrm{X}(2.2) = \mathrm{X}(2.2-2) = \mathrm{X}(0.2) = 10\,\mathrm{tri}(0.4)e^{-j4\pi(0.2)} = 6\angle -2.5133 .$$

(c) What is the numerical fundamental period of $\mathrm{Y}(F)$?

Use the property

$$\text{If } \mathrm{z}[n] = \left\{ \begin{matrix} \mathrm{x}[n/m] & , & n/m \text{ an integer} \\ 0 & , & \text{otherwise} \end{matrix} \right\} , \ \mathrm{z}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(mF)$$

$\mathrm{Y}(F)$ is a compressed version of $\mathrm{X}(F)$ and the compression factor is 4. So the value of $\mathrm{X}(F)$ at $F = 1/2$ is the same as the value of $\mathrm{Y}(F)$ at $F = 1/8$ and the fundamental period of $\mathrm{Y}(F)$ is therefore 1/4.

Alternate Solution: $\mathrm{Y}(F) = \mathrm{X}(4F) = 10\,\mathrm{tri}(8F)e^{-j16\pi F} * \delta_1(4F)$

Using the scaling property of the periodic impulse and the scaling property of convolution

$\mathrm{Y}(F) = 4 \times 10\,\mathrm{tri}(8F)e^{-j16\pi F} * (1/4)\delta_{1/4}(F) = 10\,\mathrm{tri}(8F)e^{-j16\pi F} * \delta_{1/4}(F)$

(d) Find the numerical magnitude and angle (in radians) of $\mathrm{Y}(0.55)$.

Using the fact that the fundamental period is 1/4,

$\mathrm{Y}(0.55) = \mathrm{Y}(0.55 - 1/2) = \mathrm{Y}(-0.05) = 10\,\mathrm{tri}(-8 \times 0.05)e^{j16\pi \times 0.05} = 6\angle 2.5133$

(e) Using Parseval's theorem $\sum_{n=-\infty}^{\infty}\left|\mathrm{x}[n]\right|^2 = \int_1 \left|\mathrm{X}(F)\right|^2 dF$ find the numerical signal energies of both $\mathrm{x}[n]$ and $\mathrm{y}[n]$.

$$E_{\mathrm{x}} = \int_1 \left|\mathrm{X}(F)\right|^2 dF = \int_{-1/2}^{1/2} \left|10\,\mathrm{tri}(2F)e^{-j4\pi F}\right|^2 dF = 100\int_{-1/2}^{1/2} \mathrm{tri}^2(2F)dF = 200\int_0^{1/2}(1-2F)^2\,dF$$

$$E_{\mathrm{x}} = 200\int_0^{1/2}\left(1-4F+4F^2\right)dF = 200\left[F-2F^2+4F^3/3\right]_0^{1/2} = 200\left[1/2-1/2+1/6\right] = 200/6 = 33.333\ldots$$

$$E_{\mathrm{y}} = \int_1 \left|\mathrm{Y}(F)\right|^2 dF = \int_{-1/2}^{1/2} \left|10\,\mathrm{tri}(8F)e^{-j16\pi F} * \delta_{1/4}(F)\right|^2 dF = 4\int_{-1/8}^{1/8}\left|10\,\mathrm{tri}(8F)e^{-j16\pi F}\right|^2 dF = 400\int_{-1/8}^{1/8}\mathrm{tri}^2(8F)dF = 800\int_0^{1/6}(1-8F)^2\,dF$$

$$E_{\mathrm{x}} = 800\int_0^{1/8}\left(1-16F+64F^2\right)dF = 800\left[F-8F^2+64F^3/3\right]_0^{1/8} = 800\left[1/8-1/8+64/1536\right] = 33.333\ldots$$

3. A periodic discrete-time signal is described over two fundamental periods by

n	$\cdots$	–2	–1	0	1	2	3	4	5	$\cdots$
$\mathrm{x}[n]$	$\cdots$	b	c	d	a	b	c	d	a	$\cdots$

The DFT of $\mathrm{x}[n]$ is $\mathrm{X}[k] = \sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N}$. Fill in the table for $\mathrm{X}[k]$ below with expressions for the real and imaginary parts of $\mathrm{X}[k]$ written as linear combinations of a, b, c and d.

$$\mathrm{X}[k] = \sum_{n=0}^{N-1}\mathrm{x}[n]e^{-j2\pi kn/N} = \sum_{n=0}^{2}\mathrm{x}[n]e^{-j2\pi kn/3}$$

$$\mathrm{X}[0] = d+a+b+c+j0$$
$$\mathrm{X}[1] = d-ja-b+jc = d-b+j(c-a)$$
$$\mathrm{X}[2] = d-a+b-c+j0$$
$$\mathrm{X}[3] = d-b+j(a-c)$$

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1. Find the numerical values of the constants.

(a) $8(\mathrm{u}[n+4]-\mathrm{u}[n-2])*\delta_8[n] \xleftrightarrow[8]{\mathcal{DFT}} Ae^{bk}\,\mathrm{drcl}(ck,D)$ (k is harmonic number)

Using $(\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1])*\delta_N[n] \xleftrightarrow[mN]{\mathcal{DFT}} \dfrac{e^{-j\pi k(n_1+n_0)/mN}}{e^{-j\pi k/mN}}(n_1-n_0)m\,\mathrm{drcl}(k/mN,n_1-n_0)\delta_m[k]$

with $m=1$, $n_0=-4$, $n_1=2$, $N=8$

$(\mathrm{u}[n+4]-\mathrm{u}[n-2])*\delta_8[n] \xleftrightarrow[8]{\mathcal{DFT}} 6e^{j3\pi k/8}\,\mathrm{drcl}(k/8,6)$

$8(\mathrm{u}[n+4]-\mathrm{u}[n-2])*\delta_8[n] \xleftrightarrow[8]{\mathcal{DFT}} 48e^{j3\pi k/8}\,\mathrm{drcl}(k/8,6)$

$A=48$, $b=j3\pi/8$, $c=1/8$, $D=6$

(b) $8(\mathrm{u}[n+4]-\mathrm{u}[n-2]) \xleftrightarrow{\mathcal{F}} Ae^{b\Omega}\,\mathrm{drcl}(c\Omega,D)$

Using $\mathrm{u}[n-n_0]-\mathrm{u}[n-n_1] \xleftrightarrow{\mathcal{F}} \dfrac{e^{-j\pi F(n_0+n_1)}}{e^{-j\pi F}}(n_1-n_0)\mathrm{drcl}(F,n_1-n_0)$ with $n_0=-4$, $n_1=2$ and $\Omega=2\pi F$

$\mathrm{u}[n+4]-\mathrm{u}[n-2] \xleftrightarrow{\mathcal{F}} 6e^{j3\Omega/2}\,\mathrm{drcl}(\Omega/2\pi,6)$

$8(\mathrm{u}[n+4]-\mathrm{u}[n-2]) \xleftrightarrow{\mathcal{F}} 48e^{j3\Omega/2}\,\mathrm{drcl}(\Omega/2\pi,6)$

$A=48$, $b=j3/2$, $c=1/2\pi$, $D=6$

2. Let $\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F) = 3\,\mathrm{tri}(2F)e^{j2\pi F} * \delta_1(F)$, a phase-shifted triangle in the range $-1/2 < F < 1/2$ that repeats that pattern periodically, with fundamental period one. Also let

$$\mathrm{y}[n] = \begin{cases} \mathrm{x}[n/5] & , \text{ if } n/5 \text{ is an integer} \\ 0 & , \text{ if } n/5 \text{ is not an integer} \end{cases}$$

and let $\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{Y}(F)$.

(a) Find the numerical magnitude and angle (in radians) of $\mathrm{X}(0.3)$. $\mathrm{X}(0.3) = 3\,\mathrm{tri}(0.6)e^{j2\pi(0.3)} = 1.2\angle 1.885$

(b) Find the numerical magnitude and angle (in radians) of $\mathrm{X}(2.2)$.

Taking advantage of the periodicity of $\mathrm{X}(F)$

$$\mathrm{X}(2.2) = \mathrm{X}(2.2-2) = \mathrm{X}(0.2) = 3\,\mathrm{tri}(0.4)e^{-j2\pi(0.2)} = 1.8\angle 1.2566 .$$

(c) What is the numerical fundamental period of $\mathrm{Y}(F)$?
Use the property

$$\text{If } \mathrm{z}[n] = \left\{ \begin{matrix} \mathrm{x}[n/m] & , & n/m \text{ an integer} \\ 0 & , & \text{otherwise} \end{matrix} \right\} , \quad \mathrm{z}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(mF)$$

$\mathrm{Y}(F)$ is a compressed version of $\mathrm{X}(F)$ and the compression factor is 5. So the value of $\mathrm{X}(F)$ at $F = 1/2$ is the same as the value of $\mathrm{Y}(F)$ at $F = 1/10$ and the fundamental period of $\mathrm{Y}(F)$ is therefore 1/5.

Alternate Solution: $\mathrm{Y}(F) = \mathrm{X}(5F) = 3\,\mathrm{tri}(10F)e^{j10\pi F} * \delta_1(5F)$

Using the scaling property of the periodic impulse and the scaling property of convolution

$$\mathrm{Y}(F) = 5 \times 3\,\mathrm{tri}(10F)e^{j10\pi F} * (1/5)\delta_{1/5}(F) = 3\,\mathrm{tri}(10F)e^{j10\pi F} * \delta_{1/5}(F)$$

(d) Find the numerical magnitude and angle (in radians) of $\mathrm{Y}(0.55)$.

Using the fact that the fundamental period is 1/5,

$$\mathrm{Y}(0.55) = \mathrm{Y}(0.55 - 3/5) = \mathrm{Y}(-0.05) = 3\,\mathrm{tri}(-10 \times 0.05)e^{-j10\pi \times 0.05} = 1.5\angle -1.5708$$

(e) Using Parseval's theorem $\sum_{n=-\infty}^{\infty}\left|x\left[n\right]\right|^2 = \int_1 \left|X\left(F\right)\right|^2 dF$ find the numerical signal energies of both $x[n]$ and $y[n]$.

$$E_x = \int_1 \left|X(F)\right|^2 dF = \int_{-1/2}^{1/2}\left|3\,\text{tri}(2F)e^{j2\pi F}\right|^2 dF = 9\int_{-1/2}^{1/2}\text{tri}^2(2F)dF = 18\int_0^{1/2}(1-2F)^2\,dF$$

$$E_x = 18\int_0^{1/2}\left(1-4F+4F^2\right)dF = 18\left[F-2F^2+4F^3/3\right]_0^{1/2} = 18\left[1/2-1/2+1/6\right]=3$$

$$E_y = \int_1 \left|Y(F)\right|^2 dF = \int_{-1/2}^{1/2}\left|3\,\text{tri}(10F)e^{j10\pi F} * \delta_{1/5}(F)\right|^2 dF = 5\int_{-1/10}^{1/10}\left|3\,\text{tri}(10F)e^{j10\pi F}\right|^2 dF = 45\int_{-1/10}^{1/10}\text{tri}^2(10F)dF = 90\int_0^{1/10}(1-10F)^2\,dF$$

$$E_x = 90\int_0^{1/10}\left(1-20F+100F^2\right)dF = 90\left[F-10F^2+100F^3/3\right]_0^{1/10} = 90\left[1/10-1/10+100/3000\right]=3$$

3. A periodic discrete-time signal is described over two fundamental periods by

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$x[n]$	$\cdots$	d	a	b	c	d	a	b	c	$\cdots$

The DFT of $x[n]$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$. Fill in the table for $X[k]$ below with expressions for the real and imaginary parts of $X[k]$ written as linear combinations of a, b, c and d.

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} = \sum_{n=0}^{2} x[n]e^{-j2\pi kn/3}$$

$$X[0] = b+c+d+a+j0$$
$$X[1] = b-jc-d+ja = b-d+j(a-c)$$
$$X[2] = b-c+d-a+j0$$
$$X[3] = b-d+j(c-a)$$