Solution of ECE 316 Final Examination Su10

1. Find the numerical values of the constants in these bilateral Laplace transforms of the form $\frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$.

(a) $3e^{-8(t-1)}\,\mathrm{u}(t-1) \xleftrightarrow{\mathcal{L}} \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$

$$3e^{-8(t-1)}\,\mathrm{u}(t-1) \xleftrightarrow{\mathcal{L}} 3\frac{e^{-s}}{s+8}$$

(b) $4\cos(32\pi t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$

$$4\cos(32\pi t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{4s}{s^2 + (32\pi)^2}$$

(c) $4e^{t+2}\sin(32\pi(t+2))\,\mathrm{u}(t+2) \xleftrightarrow{\mathcal{L}} \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$

$$4e^{t+2}\sin(32\pi(t+2))\,\mathrm{u}(t+2) \xleftrightarrow{\mathcal{L}} \frac{128\pi}{(s-1)^2 + (32\pi)^2} e^{2s}$$

2. Find the numerical values of the constants in these bilateral z transforms of the form $\frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$.

(a) $14(-0.3)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$

$$14(-0.3)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{14z}{z+0.3}$$

(b) $6(0.8)^{n-1} \cos(\pi(n-1)/12)\mathrm{u}[n-1] \xleftrightarrow{\mathcal{Z}} \frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$

$$6(0.8)^{n-1} \cos(\pi(n-1)/12)\mathrm{u}[n-1] \xleftrightarrow{\mathcal{Z}} 6\frac{z - 0.8\cos(\pi/12)}{z^2 - 1.6\cos(\pi/12)z + (0.8)^2} = \frac{6z - 4.6362}{z^2 - 1.5455z + 0.64}$$

3. A continuous-time signal $\mathrm{x}(t) = 10\,\mathrm{sinc}(25t)$ is sampled at $f_s = 20$ samples/second to form $\mathrm{x}[n]$ and $\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{j\Omega})$.

(a) (7 pts) Find an expression for $\mathrm{X}(e^{j\Omega})$ of the form $\{\text{Function of } \Omega\} * \delta_{2\pi}(\Omega)$.
(Remember the scaling property of convolution, if $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ then $\mathrm{y}(at) = |a|\mathrm{x}(at) * \mathrm{h}(at)$ and the scaling property of the periodic impulse, $\delta_T(at) = (1/|a|)\delta_{T/a}(t)$.)

$$\mathrm{x}[n] = 10\,\mathrm{sinc}(1.25n) \xleftrightarrow{\mathcal{F}} 8\,\mathrm{rect}(0.8F) * \delta_1(F)$$

$$\mathrm{x}[n] = 10\,\mathrm{sinc}(1.25n) \xleftrightarrow{\mathcal{F}} (8/2\pi)\mathrm{rect}(0.8\Omega/2\pi) * \delta_1(\Omega/2\pi) = 8\,\mathrm{rect}(0.1273\Omega) * \delta_{2\pi}(\Omega)$$

or

$$\mathrm{x}[n] = 10\,\mathrm{sinc}(1.25n) \xleftrightarrow{\mathcal{F}} 8\,\mathrm{rect}(\Omega/2.5\pi) * \delta_{2\pi}(\Omega)$$

(b) What is the maximum numerical magnitude of $\mathrm{X}(e^{j\Omega})$?

The aliases overlap slightly and the maximum magnitude of $\mathrm{X}(e^{j\Omega})$ is therefore 16.

4. The analog filter transfer function $\mathrm{H}_a(s) = \dfrac{8}{(s+10)(s+4)}$ is to be approximated by a digital filter. Find the transfer functions of the digital filter and report the numerical locations of all the finite poles and zeros of the digital filter for each design technique below.

(a) Impulse Invariant with $f_s = 40$ samples/second .

$$\mathrm{H}_a(s) = (4/3)\left(\frac{1}{s+4} - \frac{1}{s+10}\right)$$

$$\mathrm{h}_a(t) = (4/3)\left(e^{-4t} - e^{-10t}\right)\mathrm{u}(t) \Rightarrow \mathrm{h}_d[n] = (4/3)\left(e^{-4n/40} - e^{-10n/40}\right)\mathrm{u}[n] = (4/3)\left(0.9048^n - 0.7788^n\right)\mathrm{u}[n]$$

$$\mathrm{H}_d(z) = (4/3)\left(\frac{z}{z-0.9048} - \frac{z}{z-0.7788}\right)$$

$$\mathrm{H}_d(z) = (4/3)\frac{0.126z}{(z-0.9048)(z-0.7788)} = \frac{0.168z}{(z-0.9048)(z-0.7788)}$$

Poles at 0.7788 and 0.9048 , One zero at zero

(b) Step Invariant with $f_s = 40$ samples/second

$$\mathrm{H}_a(s) = \frac{8}{(s+10)(s+4)}$$

$$\mathrm{H}_{-1a}(s) = \frac{8}{s(s+10)(s+4)} = \frac{1/5}{s} + \frac{2/15}{s+10} - \frac{1/3}{s+4}$$

$$\mathrm{h}_{-1a}(t) = (1/15)\left(3 + 2e^{-10t} - 5e^{-4t}\right)\mathrm{u}(t) \Rightarrow \mathrm{h}_{-1d}[n] = (1/15)\left(3 + 2e^{-n/4} - 5e^{-n/10}\right)\mathrm{u}[n] = (1/15)\left(3 + 2(0.7788)^n - 5(0.9048)^n\right)\mathrm{u}[n]$$

$$\mathrm{H}_{-1d}(z) = (1/15)\left(\frac{3z}{z-1} + \frac{2z}{z-0.7788} - \frac{5z}{z-0.9048}\right)$$

$$\mathrm{H}_d(z) = (1/15)\left(3 + \frac{2(z-1)}{z-0.7788} - \frac{5(z-1)}{z-0.9048}\right)$$

$$\mathrm{H}_d(z) = \frac{1}{15}\frac{0.0336z + 0.02957}{(z-0.7788)(z-0.9048)} = \frac{0.00224z + 0.001972}{(z-0.7788)(z-0.9048)}$$

Poles at 0.7788 and 0.9048 , Zero at -0.8802

(c) Matched z Transform with $f_s = 60$ samples/second .

$$\mathrm{H}_a(s) = \frac{8}{(s+10)(s+4)} \Rightarrow \mathrm{H}_d(z) = \frac{8}{\left(1 - 0.8465z^{-1}\right)\left(1 - 0.9355z^{-1}\right)} = \frac{8z^2}{(z-0.8465)(z-0.9355)}$$

Poles at 0.8465 and 0.9355 , Two Zeros at 0

(d) Bilinear z Transform with $f_s = 60$ samples/second

$$\mathrm{H}_a(s) = \frac{8}{(s+10)(s+4)} \Rightarrow \mathrm{H}_d(z) = \frac{8}{\left(120\frac{z-1}{z+1} + 10\right)\left(120\frac{z-1}{z+1} + 4\right)}$$

$$\mathrm{H}_d(z) = \frac{0.0004963(z+1)^2}{z^2 - 1.782z + 0.7916}$$

Poles at 0.9355 and 0.8465 , Two Zeros at -1.

Solution of ECE 316 Final Examination Su10

1. Find the numerical values of the constants in these bilateral Laplace transforms of the form $\frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$.

(a) $7e^{-12(t-2)}\,\mathrm{u}(t-1) \xleftrightarrow{\mathcal{L}} \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$

$$7e^{-12(t-2)}\,\mathrm{u}(t-1) = 7e^{12} e^{-12(t-1)}\,\mathrm{u}(t-1) \xleftrightarrow{\mathcal{L}} 7e^{12}\frac{e^{-2s}}{s+12} = 1.1393 \times 10^6 \frac{e^{-2s}}{s+12}$$

(b) $18\cos(70\pi t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$

$$18\cos(70\pi t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{18s}{s^2 + (70\pi)^2}$$

(c) $9e^{t+4}\sin(70\pi(t+4))\,\mathrm{u}(t+4) \xleftrightarrow{\mathcal{L}} \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} e^{-st_0}$

$$9e^{t+4}\sin(70\pi(t+4))\,\mathrm{u}(t+4) \xleftrightarrow{\mathcal{L}} \frac{630\pi}{(s-1)^2 + (70\pi)^2} e^{4s}$$

2. Find the numerical values of the constants in these bilateral z transforms of the form $\frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$.

(a) $11(-0.6)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$

$$11(-0.6)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{11z}{z+0.6}$$

(b) $15(0.3)^{n-1} \cos(\pi(n-1)/16)\mathrm{u}[n-1] \xleftrightarrow{\mathcal{Z}} \frac{b_2 z^2 + b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}$

$$15(0.3)^{n-1} \cos(\pi(n-1)/16)\mathrm{u}[n-1] \xleftrightarrow{\mathcal{Z}} 15\frac{z - 0.3\cos(\pi/16)}{z^2 - 0.6\cos(\pi/16)z + (0.3)^2} = \frac{15z - 4.4135}{z^2 - 0.5885z + 0.09}$$

3. A continuous-time signal $\mathrm{x}(t) = 10\,\mathrm{sinc}(75t)$ is sampled at $f_s = 50$ samples/second to form $\mathrm{x}[n]$ and $\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(e^{j\Omega})$.

(a) (7 pts) Find an expression for $\mathrm{X}(e^{j\Omega})$ of the form $\{\text{Function of } \Omega\} * \delta_{2\pi}(\Omega)$.
(Remember the scaling property of convolution, if $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ then $\mathrm{y}(at) = |a|\mathrm{x}(at) * \mathrm{h}(at)$ and the scaling property of the periodic impulse, $\delta_T(at) = (1/|a|)\delta_{T/a}(t)$.)

$$\mathrm{x}[n] = 10\,\mathrm{sinc}(1.5n) \xleftrightarrow{\mathcal{F}} 6.667\,\mathrm{rect}(0.667F) * \delta_1(F)$$

$$\mathrm{x}[n] = 10\,\mathrm{sinc}(1.5n) \xleftrightarrow{\mathcal{F}} (6.667/2\pi)\mathrm{rect}(0.667\Omega/2\pi) * \delta_1(\Omega/2\pi) = 6.667\,\mathrm{rect}(0.1062\Omega) * \delta_{2\pi}(\Omega)$$

or

$$\mathrm{x}[n] = 10\,\mathrm{sinc}(1.5n) \xleftrightarrow{\mathcal{F}} 6.67\,\mathrm{rect}(\Omega/3\pi) * \delta_{2\pi}(\Omega)$$

(b) What is the maximum numerical magnitude of $\mathrm{X}(e^{j\Omega})$?

The aliases overlap slightly and the maximum magnitude of $\mathrm{X}(e^{j\Omega})$ is therefore 13.333.

4. The analog filter transfer function $H_a(s) = \dfrac{14}{(s+8)(s+3)}$ is to be approximated by a digital filter. Find the transfer functions of the digital filter and report the numerical locations of all the finite poles and zeros of the digital filter for each design technique below.

(a) Impulse Invariant with $f_s = 40$ samples/second.

$$H_a(s) = 2.8\left(\frac{1}{s+3} - \frac{1}{s+8}\right)$$

$$h_a(t) = 2.8\left(e^{-3t} - e^{-8t}\right)u(t) \Rightarrow h_d[n] = 2.8\left(e^{-3n/40} - e^{-8n/40}\right)u[n] = 2.8\left(0.9277^n - 0.8187^n\right)u[n]$$

$$H_d(z) = 2.8\left(\frac{z}{z-0.9277} - \frac{z}{z-0.8187}\right)$$

$$H_d(z) = 2.8\frac{0.109z}{(z-0.9277)(z-0.8187)} = \frac{0.3052z}{(z-0.9277)(z-0.8187)}$$

Poles at 0.8187 and 0.9277, One zero at zero

(b) Step Invariant with $f_s = 40$ samples/second

$$H_a(s) = \frac{14}{(s+8)(s+3)}$$

$$H_{-1a}(s) = \frac{14}{s(s+8)(s+3)} = \frac{0.5833}{s} + \frac{0.35}{s+8} - \frac{0.9333}{s+3}$$

$$h_{-1a}(t) = \left(0.5833 + 0.35e^{-8t} - 0.9333e^{-3t}\right)u(t) \Rightarrow h_{-1d}[n] = \left(0.5833 + 0.35e^{-n/5} - 0.9333e^{-3n/40}\right)u[n]$$

$$= \left(0.5833 + 0.35(0.8187)^n - 0.9333(0.9277)^n\right)u[n]$$

$$H_{-1d}(z) = \left(\frac{0.5833z}{z-1} + \frac{0.35z}{z-0.8187} - \frac{0.9333z}{z-0.9277}\right)$$

$$H_d(z) = \left(0.58333 + \frac{0.35(z-1)}{z-0.8187} - \frac{0.9333(z-1)}{z-0.9277}\right)$$

$$H_d(z) = \frac{0.004023z + 0.003623}{(z-0.8187)(z-0.9277)}$$

Poles at 0.8187 and 0.9277, Zero at -0.9007

(c) Matched z Transform with $f_s = 60$ samples/second.

$$H_a(s) = \frac{14}{(s+8)(s+3)} \Rightarrow H_d(z) = \frac{14}{\left(1-0.8752z^{-1}\right)\left(1-0.9512z^{-1}\right)} = \frac{14z^2}{(z-0.8752)(z-0.9512)}$$

Poles at 0.8752 and 0.9512, Two Zeros at 0

(d) Bilinear z Transform with $f_s = 60$ samples/second

$$H_a(s) = \frac{14}{(s+8)(s+3)} \Rightarrow H_d(z) = \frac{14}{\left(120\frac{z-1}{z+1} + 8\right)\left(120\frac{z-1}{z+1} + 3\right)}$$

$$H_d(z) = \frac{0.0008892(z+1)^2}{z^2 - 1.826z + 0.8323}$$

Poles at 0.9512 and 0.8752, Two Zeros at -1.