Solution of ECE 316 Test 3 S10

1. The loop transfer function for a discrete-time feedback system has N finite poles, all inside the unit circle of the z plane, and M finite zeros, also all inside the unit circle. It has an adjustable gain factor K.

   (a) What relationship between N and M guarantees that, at some finite positive value of K, the system will become unstable?

   If M is less than N, the system will become unstable at some finite positive value of K because at least one branch of the root locus must terminate on a zero at infinity. Also, if M is greater than N at least one pole is at infinity and the root locus will begin on it, making the system unstable. The case $M > N$ creates a non-causal system which, therefore, cannot be realized physically. So the precise answer is $N \neq M$. However, because the non-causal case is not realizable I counted as correct $N > M$. I also counted as correct $M > N$ because, strictly speaking it is correct even though the system is not realizable.

   (b) Change the description of the loop transfer function to allow some of the finite zeros to be outside the unit circle. How does this change the answer to part (a), if at all?

   If any of the finite zeros is outside the unit circle, the system will become unstable at a finite positive value of K, regardless of the values of N and M. So the criterion for instability does change.

2. A continuous-time unity-gain (tracking) feedback system has a forward-path transfer function with exactly one pole at the origin of the s plane.

   (a) Describe in as much detail as possible the steady-state error signal in response to a step excitation.

   This is a type one system and the steady-state error for step excitation will be zero.

   (b) Describe in as much detail as possible the steady-state error signal in response to a ramp excitation.

   For a type one system in response to a ramp excitation, the steady-error will be non-zero, but finite.

3. The transfer function of a continuous-time system has one finite pole in the open left half-plane at $s = s_0$ and no finite zeros. If you wanted to make its response to a step excitation approach its final value faster how would you change s_0?

   Make s_0 more negative. Some students answered "make it bigger". For real numbers, the usual mathematical interpretation of "bigger" is "greater than" or "more positive" and that is not correct.

4. The transfer function of a discrete-time system has two complex-conjugate finite poles inside the open unit circle and two zeros at $z = 0$. Its unit sequence response overshoots and rings before settling to its final value.

(a) If you want to increase the rate at which it rings but not change the settling time how would you move the poles?

Keep their distance from the origin the same but make their angular separation from the positive real axis larger.

(b) If you want to decrease the settling time but keep the ringing rate the same how would you move the poles?

Keep the angles of the poles the same but decrease their distance from the origin.

These answers can be seen in the inverse z transform of $\mathrm{H}(z) = \dfrac{z^2}{z^2 + 2\alpha\cos(\Omega_0)z + \alpha^2}$ which has two zeros at $z = 0$ and two finite complex-conjugate poles. The inverse transform has terms of the form

$$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \text{ and } \alpha^n \sin(\Omega_0 n)\mathrm{u}[n]$$

α is the radius out to the pole location and Ω_0 is the angle to the pole from the positive real axis. So a larger α creates a more slowly decaying exponentially-damped response and a larger Ω_0 creates a more rapidly oscillating cosine or sine.

5. A continuous-time system has a transfer function $\mathrm{H}_1(s) = \dfrac{1}{s+a}$ and $a < 0$ making it unstable. In an effort to stabilize the system, feedback with transfer function $\mathrm{H}_2(s) = K$ is used (K is a constant).

(a) What is the location of the pole of the feedback system?

(b) What relation between a and K makes the system stable?

The system transfer function is $\mathrm{H}(s) = \dfrac{\frac{1}{s+a}}{1 + \frac{K}{s+a}} = \dfrac{1}{s+a+K}$. The pole is at $s = -a - K$. For stability we want $-a - K < 0$. Therefore, if $-a - K < 0$ that means that $K > -a$ for stability.

6. The loop transfer function of a continuous-time feedback system is $\mathrm{T}(s) = \dfrac{K(s-1)}{s(s^2+3)}$. How many of its branches will approach infinity and at what numerical angles (in radians) from the positive real axis (counter-clockwise being a positive angle)?

There are three finite poles and one finite zero. Therefore two branches will approach infinity and their angles will be $\pm\pi/2$ or $\pi/2$ and $3\pi/2$.

7. The loop transfer function of a discrete-time feedback system is $\mathrm{T}(z) = \dfrac{z(z+0.2)}{(z-0.1)(z-0.8)(z^2+0.6)}$. What regions of the real axis in the z plane are part of the root locus?

There are zeros on the real axis at $z = 0$ and at $z = -0.2$ and there are poles on the real axis at $z = 0.1$ and at $z = 0.8$. The allowed regions are the ones for which the sum of the number of zeros and/or poles lying to the right is an odd number. Therefore the allowed regions are $0.1 < z < 0.8$ and $-0.2 < z < 0$.

8. A continuous-time feedback system has a forward-path transfer function $\mathrm{H}_1 = \dfrac{K}{(s+2)(s+5)}$ and a feedback-path transfer function $\mathrm{H}_2 = \dfrac{s+a}{(s+11)}$. What range of numerical values of a makes this system stable for all finite positive values of K? (The root locus will have two branches that break out of the real axis in the left half-plane and approach two vertical asymptotes. If the asymptotes are in the left half-plane those two branches will never cross into the right half-plane.)

The root locus will have two branches approaching infinity.

Case 1. If the zero is to the left of -5 these branches will break out of the real axis between $s=-2$ and $s=-5$ and the pole at $s=-11$ will approach the zero along the real axis.

Case 2. If the zero is to the right of -5, these branches will break out of the real axis between $s=-5$ and $s=-11$ and the pole at $s=-2$ will approach the zero along the real axis. If the zero is in the right half-plane, at some finite positive value of K the system will become unstable. So, for stability, a must be a positive number.

We want the centroid of the asymptotes to lie in the left half-plane. The centroid lies at the sum of the finite poles minus the sum of the finite zeros divided by m, the number of finite poles minus the number of finite zeros. So the centroid is at or $\dfrac{-2-5-11+a}{2}$. To make sure the centroid is in the left half-plane we want $\dfrac{-2-5-11+a}{2}<0$ implying that $a<18$. Therefore, overall, the range of a for stability is $0<a<18$ placing the zero between -18 and zero.

9. A continuous-time feedback system has a forward-path transfer function $\mathrm{H}_1(s) = \dfrac{10}{s+12}$ and a feedback-path transfer function $\mathrm{H}_2(s) = \dfrac{1}{s+a}$, $a>0$. What range of real values of a would make the step response overshoot and ring before settling while also keeping the system stable?

$$\mathrm{H}(s) = \frac{\dfrac{10}{s+12}}{1+\dfrac{10}{s+12}\dfrac{1}{s+a}} = \frac{10(s+a)}{s^2+(12+a)s+12a+10}$$

$$s = \frac{-(12+a)\pm\sqrt{(12+a)^2-4(12a+10)}}{2} = \frac{-(12+a)\pm\sqrt{a^2-24a+104}}{2}$$

To overshoot and ring the poles must occur in complex-conjugate pairs. This will occur if $a^2-24a+104<0$. Solving $a^2-24a+104=0$ we get 18.3246 and 5.6754. If a is less than 18.3246 and greater than 5.6754, we get complex-conjugate poles. In that range the real part of s is guaranteed negative so the system is stable. So the range of a values is $5.6754<a<18.3246$.

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1. A continuous-time system has a transfer function $H_1(s) = \frac{1}{s+a}$ and $a < 0$ making it unstable. In an effort to stabilize the system, feedback with transfer function $H_2(s) = K$ is used (*K* is a constant).

   (a) What is the location of the pole of the feedback system?

   (b) What relation between *a* and *K* makes the system stable?

The system transfer function is $H(s) = \dfrac{\frac{1}{s+a}}{1+\frac{K}{s+a}} = \dfrac{1}{s+a+K}$. The pole is at $s = -a - K$. For stability we want $-a - K < 0$. Therefore, if $-a - K < 0$ that means that $K > -a$ for stability.

2. The loop transfer function of a continuous-time feedback system is $T(s) = \dfrac{K(s-1)}{s(s^2+3)}$. How many of its branches will approach infinity and at what numerical angles (in radians) from the positive real axis (counter-clockwise being a positive angle)?

There are three finite poles and one finite zero. Therefore two branches will approach infinity and their angles will be $\pm\pi/2$ or $\pi/2$ and $3\pi/2$.

3. The loop transfer function for a discrete-time feedback system has *N* finite poles, all inside the unit circle of the *z* plane, and *M* finite zeros, also all inside the unit circle. It has an adjustable gain factor *K*.

   (a) What relationship between *N* and *M* guarantees that, at some finite positive value of *K*, the system will become unstable?

   If *M* is less than *N*, the system will become unstable at some finite positive value of *K* because at least one branch of the root locus must terminate on a zero at infinity. Also, if *M* is greater than *N* at least one pole is at infinity and the root locus will begin on it, making the system unstable. The case $M > N$ creates a non-causal system which, therefore, cannot be realized physically. So the precise answer is $N \neq M$. However, because the non-causal case is not realizable I counted as correct $N > M$. I also counted as correct $M > N$ because, strictly speaking it is correct even though the system is not realizable.

   (b) Change the description of the loop transfer function to allow some of the finite zeros to be outside the unit circle. How does this change the answer to part (a), if at all?

   If any of the finite zeros is outside the unit circle, the system will become unstable at a finite positive value of *K*, regardless of the values of *N* and *M*. So the criterion for instability does change.

4. A continuous-time unity-gain (tracking) feedback system has a forward-path transfer function with exactly one pole at the origin of the s plane.

   (a) Describe in as much detail as possible the steady-state error signal in response to a step excitation.

   This is a type one system and the steady-state error for step excitation will be zero.

   (b) Describe in as much detail as possible the steady-state error signal in response to a ramp excitation.

   For a type one system in response to a ramp excitation, the steady-error will be non-zero, but finite.

5. The transfer function of a continuous-time system has one finite pole in the open left half-plane at $s = s_0$ and no finite zeros. If you wanted to make its response to a step excitation approach its final value faster how would you change s_0 ?

   Make s_0 more negative. Some students answered "make it bigger". For real numbers, the usual mathematical interpretation of "bigger" is "greater than" or "more positive" and that is not correct.

6. The transfer function of a discrete-time system has two complex-conjugate finite poles inside the open unit circle and two zeros at $z = 0$. Its unit sequence response overshoots and rings before settling to its final value.

   (a) If you want to increase the rate at which it rings but not change the settling time how would you move the poles?

   Keep their distance from the origin the same but make their angular separation from the positive real axis larger.

   (b) If you want to decrease the settling time but keep the ringing rate the same how would you move the poles?

   Keep the angles of the poles the same but decrease their distance from the origin.

   These answers can be seen in the inverse z transform of $\mathrm{H}(z) = \dfrac{z^2}{z^2 + 2\alpha\cos(\Omega_0)z + \alpha^2}$ which has two zeros at $z = 0$ and two finite complex-conjugate poles. The inverse transform has terms of the form

   $$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \text{ and } \alpha^n \sin(\Omega_0 n)\mathrm{u}[n]$$

   α is the radius out to the pole location and Ω_0 is the angle to the pole from the positive real axis. So a larger α creates a more slowly decaying exponentially-damped response and a larger Ω_0 creates a more rapidly oscillating cosine or sine.

7. The loop transfer function of a discrete-time feedback system is $\mathrm{T}(z) = \dfrac{z(z-0.3)}{(z-0.2)(z+0.5)(z^2+0.6)}$. What regions of the real axis in the z plane are part of the root locus?

   There are zeros on the real axis at $z = 0$ and at $z = 0.3$ and there are poles on the real axis at $z = 0.2$ and at $z = -0.5$. The allowed regions are the ones for which the sum of the number of zeros and/or poles lying to the right is an odd number. Therefore the allowed regions are $0.2 < z < 0.3$ and $-0.5 < z < 0$.

8. A continuous-time feedback system has a forward-path transfer function $\mathrm{H}_1 = \dfrac{K}{(s+2)(s+5)}$ and a feedback-path transfer function $\mathrm{H}_2 = \dfrac{s+a}{(s+7)}$. What range of numerical values of a makes this system stable for all finite positive values of K? (The root locus will have two branches that break out of the real axis in the left half-plane and approach two vertical asymptotes. If the asymptotes are in the left half-plane those two branches will never cross into the right half-plane.)

The root locus will have two branches approaching infinity.

Case 1. If the zero is to the left of -5 these branches will break out of the real axis between $s=-2$ and $s=-5$ and the pole at $s=-7$ will approach the zero along the real axis.

Case 2. If the zero is to the right of -5, these branches will break out of the real axis between $s=-5$ and $s=-7$ and the pole at $s=-2$ will approach the zero along the real axis. If the zero is in the right half-plane, at some finite positive value of K the system will become unstable. So, for stability, a must be a positive number.

We want the centroid of the asymptotes to lie in the left half-plane. The centroid lies at the sum of the finite poles minus the sum of the finite zeros divided by m, the number of finite poles minus the number of finite zeros. So the centroid is at or $\dfrac{-2-5-7+a}{2}$. To make sure the centroid is in the left half-plane we want $\dfrac{-2-5-7+a}{2} < 0$ implying that $a < 14$. Therefore, overall, the range of a for stability is $0 < a < 14$ placing the zero between -14 and zero.

9. A continuous-time feedback system has a forward-path transfer function $\mathrm{H}_1(s) = \dfrac{4}{s+8}$ and a feedback- path transfer function $\mathrm{H}_2(s) = \dfrac{1}{s+a}$, $a > 0$. What range of real values of a would make the step response overshoot and ring before settling while also keeping the system stable?

$$\mathrm{H}(s) = \frac{\dfrac{4}{s+8}}{1+\dfrac{4}{s+8}\dfrac{1}{s+a}} = \frac{4(s+a)}{s^2+(8+a)s+8a+4}$$

$$s = \frac{-(8+a) \pm \sqrt{(8+a)^2 - 4(8a+4)}}{2} = \frac{-(8+a) \pm \sqrt{a^2-16a+48}}{2}$$

To overshoot and ring the poles must occur in complex conjugate pairs. This will occur if $a^2-16a+48<0$. Solving $a^2-16a+48=0$ we get 12 and 4 . If a is less than 12 and greater than 4, we get complex-conjugate poles. If a were less than -8, the poles would be in the right half-plane. So the range of a values is $4 < a < 12$.

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   (a) What is the location of the pole of the feedback system? Pole at $s =$ ____________

   (b) What relation between *a* and *K* makes the system stable?

The system transfer function is $\mathrm{H}(s) = \dfrac{\frac{1}{s+a}}{1+\frac{K}{s+a}} = \dfrac{1}{s+a+K}$. The pole is at $s = -a - K$. For stability we want $-a - K < 0$. Therefore, if $-a - K < 0$ that means that $K > -a$ for stability.

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   For a type one system in response to a ramp excitation, the steady-error will be non-zero, but finite.

7. The loop transfer function of a discrete-time feedback system is $\mathrm{T}(z) = \dfrac{z(z+0.7)}{(z-0.9)(z+0.1)(z^2+0.6)}$. What regions of the real axis in the z plane are part of the root locus?

   There are zeros on the real axis at $z = 0$ and at $z = -0.7$ and there are poles on the real axis at $z = -0.1$ and at $z = 0.9$. The allowed regions are the ones for which the sum of the number of zeros and/or poles lying to the right is an odd number. Therefore the allowed regions are $0 < z < 0.9$ and $-0.7 < z < -0.1$.

8. A continuous-time feedback system has a forward-path transfer function $H_1 = \frac{K}{(s+8)(s+5)}$ and a feedback-path transfer function $H_2 = \frac{s+a}{(s+11)}$. What range of numerical values of a makes this system stable for all finite positive values of K? (The root locus will have two branches that break out of the real axis in the left half-plane and approach two vertical asymptotes. If the asymptotes are in the left half-plane those two branches will never cross into the right half-plane.)

The root locus will have two branches approaching infinity.

Case 1. If the zero is to the left of -8 these branches will break out of the real axis between $s=-5$ and $s=-8$ and the pole at $s=-11$ will approach the zero along the real axis.

Case 2. If the zero is to the right of -8, these branches will break out of the real axis between $s=-8$ and $s=-11$ and the pole at $s=-5$ will approach the zero along the real axis. If the zero is in the right half-plane, at some finite positive value of K the system will become unstable. So, for stability, a must be a positive number.

We want the centroid of the asymptotes to lie in the left half-plane. The centroid lies at the sum of the finite poles minus the sum of the finite zeros divided by m, the number of finite poles minus the number of finite zeros. So the centroid is at or $\frac{-5-8-11+a}{2}$. To make sure the centroid is in the left half-plane we want $\frac{-5-8-11+a}{2} < 0$ implying that $a < 24$. Therefore, overall, the range of a for stability is $0 < a < 24$ placing the zero between -24 and zero.

9. A continuous-time feedback system has a forward-path transfer function $H_1(s) = \frac{7}{s+3}$ and a feedback- path transfer function $H_2(s) = \frac{1}{s+a}$, $a > 0$. What range of real values of a would make the step response overshoot and ring before settling while also keeping the system stable?

$$H(s) = \frac{\frac{7}{s+3}}{1+\frac{7}{s+3}\frac{1}{s+a}} = \frac{10(s+a)}{s^2+(3+a)s+3a+7}$$

$$s = \frac{-(3+a) \pm \sqrt{(3+a)^2 - 4(3a+7)}}{2} = \frac{-(3+a) \pm \sqrt{a^2-6a-19}}{2}$$

To overshoot and ring the poles must occur in complex conjugate pairs. This will occur if $a^2-6a-19<0$. Solving $a^2-6a-19=0$ we get 8.2915 and -2.2915 . If a is less than 8.2915 and greater than -2.2915, we get complex-conjugate poles. If a were less than -3, the poles would be in the right half-plane. So the range of a values is $-2.2915 < a < 8.2915$. But to observe the restriction above, $a > 0$, that range should be $0 < a < 8.2915$.