Solution to ECE Test #2 Su06

1. The inverse Laplace transform of $\mathrm{X}(s) = \dfrac{2s(s+5)}{s^2+9}$ can be expressed in the form

$$\mathrm{x}(t) = A\delta(t) + B\left[\cos(at) + C\sin(bt)\right]\mathrm{u}(t).$$

Find numerical values for *A*, *B*, *C*, *a* and *b*.

$A = 2$, $B = 10$, $C = -0.6$, $a = 3$, $b = 3$.

$$\begin{array}{r} 2 \\ s^2+9 \overline{\big)\, 2s^2 + 10s} \\ \underline{2s^2 + 18} \\ 10s - 18 \end{array} \Rightarrow \mathrm{X}(s) = 2 + \frac{10s-18}{s^2+9} = 2 + 10\left(\frac{s}{s^2+9} - \frac{1.8}{3}\frac{3}{s^2+9}\right)$$

$$\mathrm{x}(t) = 2\delta(t) + 10\left[\cos(3t) - 0.6\sin(3t)\right]\mathrm{u}(t)$$

2. The inverse Laplace transform of $\mathrm{X}(s)=\dfrac{3s-2}{(s+4)^2}$ can be expressed in the form

$$\mathrm{x}(t)=A(1+Bt)e^{at}\,\mathrm{u}(t)\,.$$

Find numerical values for *A*, *B* and *a*.

$A=3$, $B=-14/3$, $a=-4$.

$$\mathrm{X}(s)=\frac{-14}{(s+4)^2}+\frac{3}{s+4}\Rightarrow \mathrm{x}(t)=\left(3e^{-4t}-14te^{-4t}\right)\mathrm{u}(t)=(3-14t)e^{-4t}\,\mathrm{u}(t)$$

$$\mathrm{x}(t)=3\big(1-(14/3)t\big)e^{-4t}\,\mathrm{u}(t)$$

Alternate Solution:

Start with

$$3\mathrm{u}(t)-14t\,\mathrm{u}(t)\xleftrightarrow{\mathcal{L}}\frac{3}{s}-\frac{14}{s^2}=\frac{3s-14}{s^2}$$

Let $s\to s+4$. Then, using the complex-frequency shifting property,

$$\mathrm{x}(t)=e^{-4t}\left[3\mathrm{u}(t)-14t\,\mathrm{u}(t)\right]=3\big(1-(14/3)t\big)e^{-t}\,\mathrm{u}(t)\xleftrightarrow{\mathcal{L}}\frac{3s-2}{(s+4)^2}=\mathrm{X}(s)$$

Second Alternate Solution:

$$\mathrm{X}(s)=\frac{3s-2}{(s+4)^2}=\frac{3s}{(s+4)^2}-\frac{2}{(s+4)^2}$$

$$\frac{d}{dt}\left(3te^{-4t}\,\mathrm{u}(t)\right)+\left[3te^{-4t}\,\mathrm{u}(t)\right]_{t=0^-}-2te^{-4t}\,\mathrm{u}(t)\xleftrightarrow{\mathcal{L}}\frac{3s}{(s+4)^2}-\frac{2}{(s+4)^2}$$

$$3\left[te^{-4t}\delta(t)+\mathrm{u}(t)\left(-4te^{-4t}+e^{-4t}\right)\right]-2te^{-4t}\,\mathrm{u}(t)\xleftrightarrow{\mathcal{L}}\frac{3s}{(s+4)^2}-\frac{2}{(s+4)^2}$$

$$3\big(1-(14/3)t\big)\mathrm{u}(t)e^{-4t}\xleftrightarrow{\mathcal{L}}\frac{3s}{(s+4)^2}-\frac{2}{(s+4)^2}$$

3. Given the Laplace-transform pair

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{L}} \frac{3s^2+9s-14}{s(s^2+2s+10)}$$

and the fact that $\mathrm{x}(t)$ is continuous at $t=0$,

(a) Find the Laplace transform of $\frac{d}{dt}(\mathrm{x}(t-2))$.

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{L}} s\frac{3s^2+9s-14}{s(s^2+2s+10)} - \mathrm{x}(0^-)$$

$\mathrm{x}(0^-) = \mathrm{x}(0^+)$ (from continuity at $t=0$)

From the initial-value theorem

$$\mathrm{x}(0^+) = \lim_{s\to\infty} s\frac{3s^2+9s-14}{s(s^2+2s+10)} = 3$$

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{L}} \frac{3s^2+9s-14}{s^2+2s+10} - 3$$

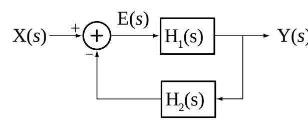
$$\frac{d}{dt}(\mathrm{x}(t-2)) \xleftrightarrow{\mathcal{L}} \left[\frac{3s^2+9s-14}{s^2+2s+10} - 3\right]e^{-2s}$$

(b) Find the numerical value of $\lim_{t\to\infty}\mathrm{x}(t)$ if it exists. If it does not exist explain how you know that.

Poles of $s\,\mathrm{X}(s)$ are at $-1 \pm j3$ in the open left half-plane. Therefore the final-value theorem applies.

$$\lim_{t\to\infty}\mathrm{x}(t) \xleftrightarrow{\mathcal{L}} \lim_{s\to 0} s\frac{3s^2+9s-14}{s(s^2+2s+10)} = -1.4$$

4. In the standard feedback configuration (illustrated below), $\mathrm{H}_1(s) = \frac{K}{s-1}$ and $\mathrm{H}_2(s) = \frac{4}{s+5}$. What numerical range of values of K makes the overall feedback system stable?



$$\mathrm{H}(s) = \frac{\frac{K}{s-1}}{1 + \frac{4K}{(s-1)(s+5)}} = \frac{K(s+5)}{(s-1)(s+5) + 4K}$$

For stability the poles must be in the open left half-plane. Poles are at

$$(s-1)(s+5) + 4K = 0$$

$$s^2 + 4s + 4K - 5 = 0 \Rightarrow s = \frac{-4 \pm \sqrt{16 - 16K + 20}}{2} = \frac{-4 \pm \sqrt{36 - 16K}}{2}$$

If $K > 9/4$ the poles are complex, the real parts are both -2, therefore in the open left half-plane and the system is stable.

If $K \leq 9/4$ the poles are real.

> If $36 - 16K \geq 16$ then one real pole will be not be in the open left half-plane. This translates into $16K \leq 20 \Rightarrow K \leq 5/4$.

So, overall, for stability, $K > 5/4$.