Solution to EE 503 Test #1 S03 (Version 1)

(Solutions to other versions are similar.)

1. Find the numerical values of the magnitude and angle of any two different complex numbers, z_1 and z_2 which both satisfy the equation, $\exp(z) = 1 - j$. That is, $\exp(z_1) = 1 - j$ and $\exp(z_2) = 1 - j$.

Magnitude of z_1 = 0.8585 Angle (Phase) of z_1 = -1.1552

Magnitude of z_2 = 5.5087 Angle (Phase) of z_2 = -1.5078

or

Magnitude of z_2 = 7.0771 Angle (Phase) of z_2 = -1.5218

or

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(an infinity of solutions)

$$z = \log(1 - j) = \log\left(\sqrt{2}e^{-j\frac{\pi}{4}}\right) = \ln\left(\sqrt{2}\right) + j\left(-\frac{\pi}{4} + 2n\pi\right)$$

$$z_1 = \ln\left(\sqrt{2}\right) - j\frac{\pi}{4} = 0.3466 - j0.7854 = 0.8585e^{-j1.1552}$$

$$z_2 = \ln\left(\sqrt{2}\right) + j\frac{7\pi}{4} = 0.3466 + j5.4978 = 5.5087e^{j1.5078}$$

or

$$z_2 = \ln\left(\sqrt{2}\right) - j\frac{9\pi}{4} = 0.3466 - j7.0686 = 7.0771e^{-j1.5218}$$

Alternate Solution:

$$\exp(z) = e^x\left(\cos(y) + j\sin(y)\right) = 1 - j$$

$$e^x\cos(y) + je^x\sin(y) = 1 - j$$

$$e^x\cos(y) = 1 \ , \ e^x\sin(y) = -1$$

Forming the ratio of these two equations,

$$\frac{\sin(y)}{\cos(y)} = \tan(y) = -1 \Rightarrow y = \tan^{-1}(-1) = -\frac{\pi}{4} + 2n\pi$$

(Note: $y \neq \frac{3\pi}{4} + 2n\pi$ because $e^x \cos(y) = 1$ and $e^x \sin(y) = -1$.)

Then $e^x \cos\left(-\frac{\pi}{4}\right) = 1 \Rightarrow e^x = \sqrt{2} \Rightarrow x = \ln\left(\sqrt{2}\right)$. Therefore $z = \ln\left(\sqrt{2}\right) + j\left(-\frac{\pi}{4} + 2n\pi\right)$ as in the first solution and the rest of the solution is the same.

2. If, on the branch, $0 \le \theta < 2\pi$, we choose to find the square root of $j2$ as

$$\sqrt{j2} = \sqrt{2e^{j\frac{\pi}{2}}} = \sqrt{2}e^{j\frac{\pi}{4}} = 1 + j,$$

find the numerical value of the residue of the function, $\frac{20\sqrt{z}}{z^4 + 16}$, at the pole, $\sqrt{2}(1 - j)$, on the branch, $0 \le \theta < 2\pi$, finding square roots the same way (magnitude of the square root is the positive square root of the magnitude and the angle of the square root is half the angle). Report the numerical value in either of the forms, $x + jy$ or $re^{j\theta}$.

$$\frac{20\sqrt{z}}{z^4 + 16} = \frac{20\sqrt{z}}{\left(z - 2e^{j\frac{\pi}{4}}\right)\left(z - 2e^{j\frac{3\pi}{4}}\right)\left(z - 2e^{j\frac{5\pi}{4}}\right)\left(z - 2e^{j\frac{7\pi}{4}}\right)}$$

$$\text{Residue at } z = \sqrt{2}(1 - j) = \frac{20\sqrt{2e^{j\frac{7\pi}{4}}}}{\left(2e^{j\frac{7\pi}{4}} - 2e^{j\frac{\pi}{4}}\right)\left(2e^{j\frac{7\pi}{4}} - 2e^{j\frac{3\pi}{4}}\right)\left(2e^{j\frac{7\pi}{4}} - 2e^{j\frac{5\pi}{4}}\right)}$$

Taking the square root according to the method prescribed,

$$\text{Residue at } z = \sqrt{2}(1 - j) = \frac{5\sqrt{2}}{2}\frac{e^{j\frac{7\pi}{8}}}{\left(e^{j\frac{7\pi}{4}} - e^{j\frac{\pi}{4}}\right)\left(e^{j\frac{7\pi}{4}} - e^{j\frac{3\pi}{4}}\right)\left(e^{j\frac{7\pi}{4}} - e^{j\frac{5\pi}{4}}\right)}$$

$$\text{Residue at } z = \sqrt{2}(1 - j) = \frac{5\sqrt{2}}{2}\frac{e^{j\frac{7\pi}{8}}}{-j\sqrt{2}\left(\sqrt{2} - j\sqrt{2}\right)\sqrt{2}} = -\frac{5\sqrt{2}}{j4}\frac{e^{j\frac{7\pi}{8}}}{\sqrt{2} - j\sqrt{2}} = -\frac{5\sqrt{2}}{j4}\frac{e^{j\frac{7\pi}{8}}}{2e^{j\frac{7\pi}{4}}}$$

$$\text{Residue at } z = \sqrt{2}(1 - j) = j\frac{5\sqrt{2}}{8}\frac{e^{j\frac{7\pi}{8}}}{e^{j\frac{7\pi}{4}}} = \frac{5\sqrt{2}}{8}e^{j\frac{\pi}{2}}\frac{e^{j\frac{7\pi}{8}}}{e^{j\frac{7\pi}{4}}} = \frac{5\sqrt{2}}{8}e^{-j\frac{3\pi}{8}}$$

$$\text{Residue at } z = \sqrt{2}(1 - j) = 0.8839e^{-j\frac{3\pi}{8}} = 0.3382 - j0.8166$$

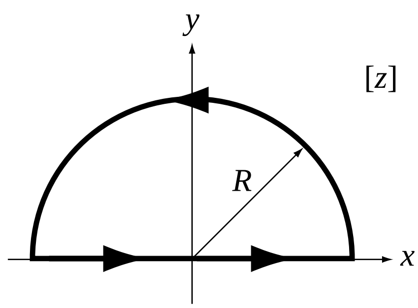
or, expressing the result with an angle on the specified branch (which is not necessary)

Residue at $z = \sqrt{2}(1 - j) = 0.8839e^{j5.1051}$.

3. Find the numerical value of the integral,

$$I = \int_0^\infty \frac{dx}{x^4+1} \,,$$

by means of a contour integral in the complex plane using the contour illustrated below as $R \to \infty$.



Since the integrand is even,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{1}{2}\int_{C,ccw} \frac{dz}{z^4+1} = \frac{1}{2}\int_{C,ccw} \frac{dz}{\left(z-\frac{1+j}{\sqrt{2}}\right)\left(z-\frac{1-j}{\sqrt{2}}\right)\left(z-\frac{-1+j}{\sqrt{2}}\right)\left(z-\frac{-1-j}{\sqrt{2}}\right)}$$

By Jordan's lemma the integral along the semicircular contour is zero, therefore

$$I = \frac{1}{2} \times j2\pi \sum \text{residues}$$

$$I = \left[\begin{array}{l} \dfrac{j\pi}{\left(\frac{1+j}{\sqrt{2}}-\frac{1-j}{\sqrt{2}}\right)\left(\frac{1+j}{\sqrt{2}}-\frac{-1+j}{\sqrt{2}}\right)\left(\frac{1+j}{\sqrt{2}}-\frac{-1-j}{\sqrt{2}}\right)} \\ +\dfrac{j\pi}{\left(\frac{-1+j}{\sqrt{2}}-\frac{1+j}{\sqrt{2}}\right)\left(\frac{-1+j}{\sqrt{2}}-\frac{1-j}{\sqrt{2}}\right)\left(\frac{-1+j}{\sqrt{2}}-\frac{-1-j}{\sqrt{2}}\right)} \end{array}\right]$$

$$I = j\pi 2\sqrt{2}\left[\frac{1}{j2\times 2\times(2+j2)} + \frac{1}{-2\times(-2+j2)\times j2}\right]$$

$$I = \frac{j\pi 2\sqrt{2}}{j4}\left(\frac{1}{2+j2} - \frac{1}{-2+j2}\right) = \frac{\pi\sqrt{2}}{2}\left(\frac{1}{2+j2} + \frac{1}{2-j2}\right)$$

$$I = \frac{\pi\sqrt{2}}{2}\left(\frac{2 - j2 + 2 + j2}{8}\right) = \frac{\pi\sqrt{2}}{4}$$