# Solution to EE 503 Test #3 S03 4/4/03

1. The CTFT of the CT signal, $x(t) = \text{tri}\left(\frac{t}{4}\right) * \text{comb}(2t)$, can be expressed as a single impulse of the form, $X(f) = A\delta(f)$. What is the numerical value of $A$?

$$X(f) = 4\,\text{sinc}^2(4f) \times \frac{1}{2}\text{comb}\left(\frac{f}{2}\right) = 2\,\text{sinc}^2(4f)\,\text{comb}\left(\frac{f}{2}\right)$$

$$X(f) = 2\,\text{sinc}^2(4f) \sum_{k=-\infty}^{\infty} \delta\left(\frac{f}{2} - k\right) = 4 \sum_{k=-\infty}^{\infty} \text{sinc}^2(4f)\delta(f - 2k)$$

All the impulses fall on nulls of the $\text{sinc}^2$ function except the one at $f = 0$ and that one has a strength of 4. Therefore $A = 4$.

2. What is the maximum numerical value of the CT function, $x(t)$, which is the inverse CTFT of $X(f) = \text{rect}\left(\frac{f}{5}\right)\text{comb}(f)$? ($x(t)$ can be expressed as a constant plus some sinusoids.)

$$X(f) = \text{rect}\left(\frac{f}{5}\right)\text{comb}(f) = \delta(f+2) + \delta(f+1) + \delta(f) + \delta(f-1) + \delta(f-2)$$

$$x(t) = 1 + 2\cos(2\pi t) + 2\cos(4\pi t)$$

Maximum value occurs when the cosines have simulatneous positive peak values, for example at $t = 0$. So the maximum value is 5.

3. Find the numerical signal energy of the DT signal, $x[n] = 25\,\text{sinc}\left(\frac{3n}{11}\right)$.

Use Parseval's theorem for DT signals.

$$X(F) = 25 \times \frac{11}{3}\text{rect}\left(\frac{11}{3}F\right) * \text{comb}(F) = \frac{275}{3}\text{rect}\left(\frac{11}{3}F\right) * \text{comb}(F)$$

The total signal energy is the area under the square of the magnitude of $X(F)$ over exactly one period. That area is

$$X(F) = \left(\frac{275}{3}\right)^2 \frac{3}{11} = \frac{226875}{99} \cong 2291.67\ .$$

4. A DT signal, $\mathrm{x}[n]$, is formed by sampling a CT signal, $\mathrm{x}(t)$, at a sampling rate of 50 Hz. The DTFT of $\mathrm{x}[n]$ is $\mathrm{X}(F) = 10\left[\mathrm{comb}\left(F - \frac{1}{4}\right) + \mathrm{comb}\left(F + \frac{1}{4}\right)\right]$. Find two different CT functions, either of which could be the function, $\mathrm{x}(t)$, that was sampled.

$$\mathrm{x}[n] = 20\cos\left(\frac{2\pi n}{4}\right)$$

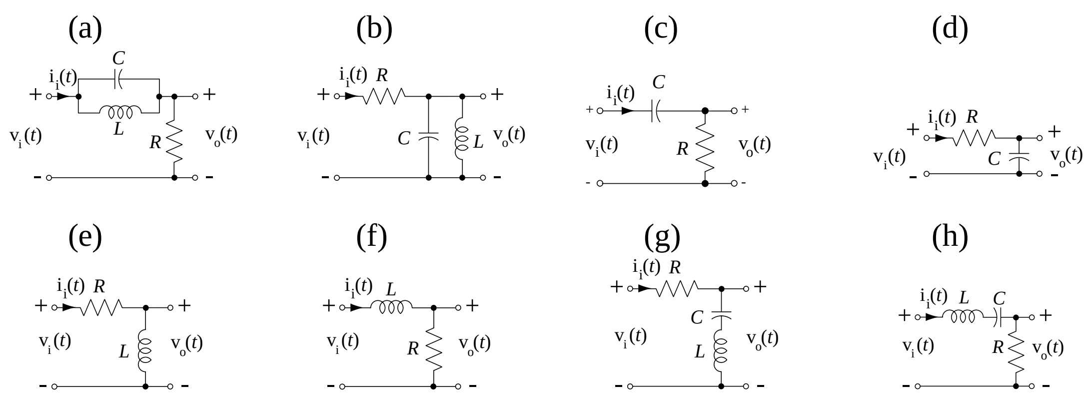
The CT signal that was sampled must be of the form, $\mathrm{x}(t) = 20\cos(2\pi f_0 t)$. When it is sampled, it is of the form, $\mathrm{x}[n] = 20\cos(2\pi f_0 n T_s) = 20\cos\left(2\pi n \frac{f_0}{f_s}\right)$. The simplest form of $\mathrm{x}(t) = 20\cos(2\pi f_0 t)$ would be found by letting $2\pi n \frac{f_0}{f_s} = \frac{2\pi n}{4}$, which implies that $\frac{f_0}{f_s} = \frac{1}{4} \Rightarrow f_0 = \frac{50}{4} = 12.5$ and $\mathrm{x}(t) = 20\cos(25\pi t)$. If the frequency is changed by any integer multiple of the sampling rate the samples do not change. Therefore, the most general form of the CT signal would be

$$\mathrm{x}(t) = 20\cos(2\pi(12.5 + 50k)t)$$

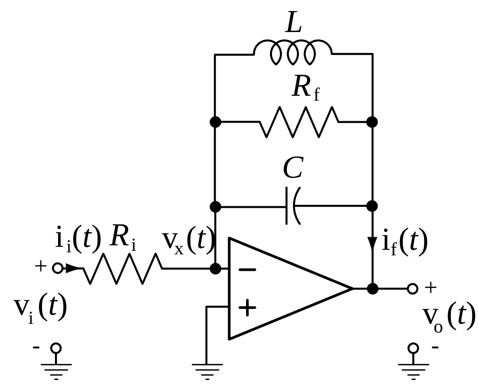
where $k$ is any integer.

5. Each practical passive filter below is an approximation to one of the four basic ideal filter types, lowpass, highpass, bandpass and bandstop. Identify the function of each filter with one of those four terms.

| | | | |
|---|---|---|---|
| (a) | Bandstop | (b) | Bandpass |
| (c) | Highpass | (d) | Lowpass |
| (e) | Highpass | (f) | Lowpass |
| (g) | Bandstop | (h) | Bandpass |



6. Let the excitation signal, $\mathrm{v}_i(t)$, in the circuit below be of the form, $\mathrm{v}_i(t) = \cos(2\pi f_0 t)$. Let the component values be $R_i = 10\ \mathrm{k}\Omega$, $R_f = 100\ \mathrm{k}\Omega$, $C = 1\ \mu\mathrm{F}$ and $L = 1\ \mathrm{mH}$. At what numerical value of $f_0$ will the response signal, $\mathrm{v}_o(t)$, have the largest possible amplitude and what will that amplitude be? (Assume the operational amplifier is ideal.)

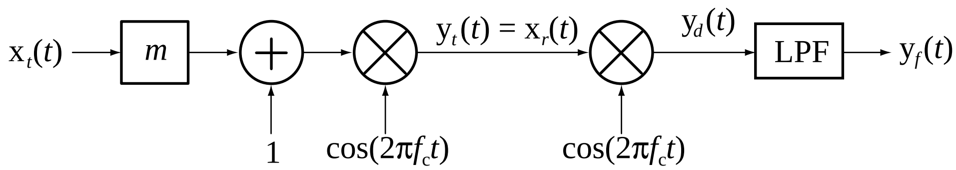



The maximum response amplitude occurs at the frequency of maximum gain magnitude. The gain is

$$\mathrm{H}(f) = -\frac{\dfrac{1}{G_f + sC + \dfrac{1}{j2\pi fL}}}{R_i} = -\frac{\dfrac{j2\pi fL}{-(2\pi f)^2 LC + j2\pi fLG_f + 1}}{R_i} = -\frac{1}{R_iC}\frac{j2\pi f}{-(2\pi f)^2 + \dfrac{j2\pi f}{R_fC} + \dfrac{1}{LC}}$$

The maximum gain occurs at resonance where $\dfrac{1}{LC} - (2\pi f)^2 = 0$. Solving for the resonant frequency, $f = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{10^{-3}\times 10^{-6}}} = 5.033\ \mathrm{kHz}$. At resonance the transfer function is $\mathrm{H}(f) = -\dfrac{R_f}{R_i} = -\dfrac{100}{10} = -10$ and, since the excitation amplitude is one, the response amplitude is 10.

7. In the system below, if $x_t(t) = 10$, $m = \frac{1}{2}$, $f_c = 100$ kHz and the lowpass filter is ideal with a cutoff frequency of 1 kHz, what is the numerical value of the response, $y_f(t)$?




$$y_t(t) = x_r(t) = 6\cos(2\pi f_c t)$$

$$y_d(t) = 6\cos^2(2\pi f_c t) = 3[1 + \cos(4\pi f_c t)] = 3\left[1 + \underbrace{\cos(4 \times 10^5 \pi t)}_{\text{removed by LPF}}\right]$$

$$y_f(t) = 3$$

8. The Nyquist sampling rate is the dividing line between undersampling and oversampling a signal. For each signal below find its Nyquist rate. If the signal is not bandlimited just write "infinite".

(a) $x(t) = -4\,\text{sinc}\left(\frac{t}{8}\right)$ Nyquist rate $= \frac{1}{8}$

$$X(f) = -32\,\text{rect}(8f) \Rightarrow f_{Nyq} = \frac{1}{8}$$

(b) $x(t) = 25\,\text{tri}\left(\frac{t-4}{2}\right)$ Nyquist rate = Infinite

$$X(f) = 50\,\text{sinc}^2(2f)e^{-j8\pi f}$$

(c) $x(t) = 3\cos(100\pi t)\sin(10{,}000\pi t)$ Nyquist rate = 10,100

$$X(f) = \frac{3}{2}[\delta(f-50) + \delta(f+50)] * \frac{j}{2}[\delta(f+5000) - \delta(f-5000)]$$

$$X(f) = j\frac{3}{4}[\delta(f-50) + \delta(f+50)] * [\delta(f+5000) - \delta(f-5000)]$$

$$X(f) = j\frac{3}{4}[\delta(f+4950) - \delta(f-5050) + \delta(f+5050) - \delta(f-4950)]$$

(d) $x(t) = -50\,\text{sinc}(20t)\cos(80\pi t)$ Nyquist rate = 100

$$X(f) = -\frac{50}{20}\text{rect}\left(\frac{f}{20}\right) * \frac{1}{2}[\delta(f-40) + \delta(f+40)]$$

$$X(f) = -\frac{5}{4}\left[\text{rect}\left(\frac{f-40}{20}\right) + \text{rect}\left(\frac{f+40}{20}\right)\right]$$

9. A signal is sampled 3 times and the samples are $\{x[0], x[1], x[2]\} = \{1, -1, -2\}$. The discrete Fourier transform (DFT) of this set of samples is $X[k]$. What is the numerical value of $X[1]$?

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{nk}{N}} \Rightarrow X[1] = \sum_{n=0}^{3-1} x[n] e^{-j2\pi\frac{n}{3}} = 1 - e^{-j\frac{2\pi}{3}} - 2e^{-j\frac{4\pi}{3}}$$

$$X[k] = 1 - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) - 2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \frac{5}{2} - j\frac{\sqrt{3}}{2} = 2.5 - j0.866$$

10. A bandlimited periodic CT signal, $x(t)$, is sampled, at a rate greater than its Nyquist rate, 4 times over exactly one fundamental period to form the DT signal, $x[n]$. The discrete Fourier transform (DFT) of that set of samples is

$$\{X[0], X[1], X[2], X[3]\} = \{2, 1-j, 0, 1+j\} \ .$$

(a) What is the numerical average value of the samples, $\{x[0], x[1], x[2], x[3]\}$?

The sum of the samples is $X[0] = 2$. There are four samples. Therefore the average of the samples is $\frac{1}{2}$.

(b) What is the numerical average signal power of $x(t)$?

Since the signal is bandlimited and sampled properly, the CTFS of the signal is

$$X[k] = \frac{1}{4}\{(1+j)\delta[k+1] + 2\delta[k] + (1-j)\delta[k-1]\} \ .$$

The average signal power is the sum of the squares of these amplitudes which is

$$\frac{2+4+2}{16} = \frac{1}{2} \ .$$