Solutions to Previous ECE 503 Examination Questions

1. A discrete-time system has a transfer function given by

$$\mathrm{H}(z) = \frac{z\left(z + \frac{1}{2}\right)}{z^2 + \frac{1}{4}} \ .$$

(a) Sketch a block diagram of the discrete-time system which has this transfer function using gain blocks and delays.

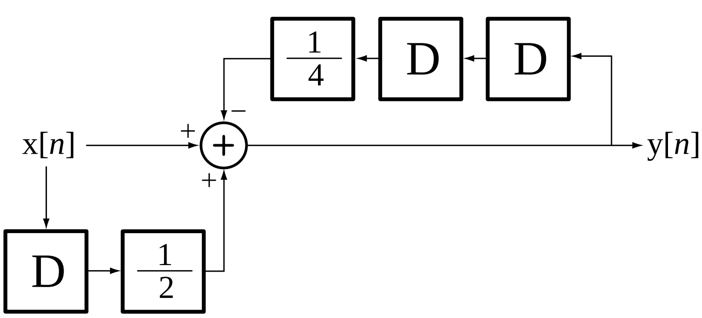
$$\frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \frac{z\left(z + \frac{1}{2}\right)}{z^2 + \frac{1}{4}}$$

$$\mathrm{Y}(z)\left(z^2 + \frac{1}{4}\right) = z\left(z + \frac{1}{2}\right)\mathrm{X}(z)$$

$$z^2\,\mathrm{Y}(z) = z^2\,\mathrm{X}(z) + \frac{z}{2}\mathrm{X}(z) - \frac{1}{4}\mathrm{Y}(z)$$

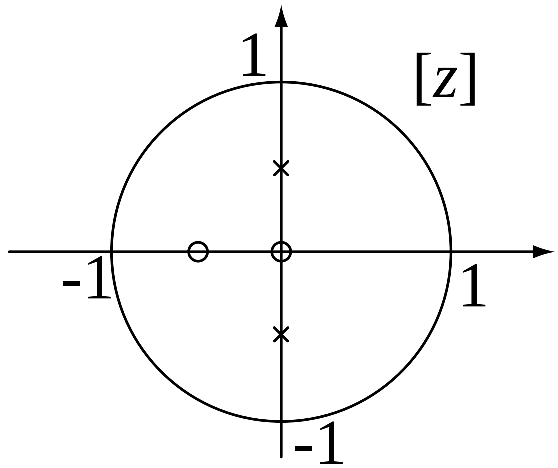
$$\mathrm{Y}(z) = \mathrm{X}(z) + \frac{z^{-1}}{2}\mathrm{X}(z) - \frac{z^{-2}}{4}\mathrm{Y}(z)$$

$$\mathrm{y}[n] = \mathrm{x}[n] + \frac{\mathrm{x}[n-1]}{2} - \frac{\mathrm{y}[n-2]}{4}$$



(b) Sketch a pole-zero diagram of this transfer function. Include scales on your sketch so that actual numerical values could reasonably be estimated from it.

$$\mathrm{H}(z) = \frac{z\left(z + \frac{1}{2}\right)}{\left(z - \frac{j}{2}\right)\left(z + \frac{j}{2}\right)}$$



(c) If the excitation of this system, $\mathrm{x}[n]$, is a discrete-time unit impulse, sketch the response, $\mathrm{y}[n]$, and identify numerically the first three values of the response. Include scales on your sketch so that actual numerical values could reasonably be estimated from it.

From (a) $$\mathrm{y}[n] = \mathrm{x}[n] + \frac{\mathrm{x}[n-1]}{2} - \frac{\mathrm{y}[n-2]}{4}$$

n	$\mathrm{x}[n]$	$\mathrm{y}[n]$
0	1	1
1	0	$\frac{1}{2}$
2	0	$-\frac{1}{4}$

$\mathrm{y}[0] = 1$

$\mathrm{y}[1] = \frac{1}{2}$

$\mathrm{y}[2] = -\frac{1}{4}$

(d) Find a closed-form solution for the impulse response of the system written in terms of real-valued functions only and check your three initial values of the impulse response against it.

$$\mathrm{H}(z)=\frac{z\left(z+\frac{1}{2}\right)}{\left(z-\frac{j}{2}\right)\left(z+\frac{j}{2}\right)}=z\left[\frac{\frac{\left(\frac{j}{2}+\frac{1}{2}\right)}{j}}{z-\frac{j}{2}}+\frac{\frac{\left(-\frac{j}{2}+\frac{1}{2}\right)}{-j}}{z+\frac{j}{2}}\right]=\frac{1}{j2}\left[\frac{(j+1)z}{z-\frac{j}{2}}+\frac{(j-1)z}{z+\frac{j}{2}}\right]$$

Using $\alpha^n\,\mathrm{u}[n]\xleftrightarrow{\mathcal{Z}}\frac{z}{z-\alpha}=\frac{1}{1-\alpha z^{-1}}$,

$$\mathrm{h}[n]=\frac{1}{j2}\left[(j+1)\left(\frac{j}{2}\right)^n+(j-1)\left(-\frac{j}{2}\right)^n\right]\mathrm{u}[n]$$

$$\mathrm{h}[n]=\frac{1}{j2}\left[(j+1)\frac{e^{j\frac{\pi}{2}n}}{2^n}+(j-1)\frac{e^{-j\frac{\pi}{2}n}}{2^n}\right]\mathrm{u}[n]=\frac{1}{j2^{n+1}}\left[(j+1)e^{j\frac{\pi}{2}n}+(j-1)e^{-j\frac{\pi}{2}n}\right]\mathrm{u}[n]$$

$$\mathrm{h}[n]=\frac{1}{2^n}\left[\frac{e^{j\frac{\pi}{2}n}+e^{-j\frac{\pi}{2}n}}{2}+\frac{e^{j\frac{\pi}{2}n}-e^{-j\frac{\pi}{2}n}}{j2}\right]\mathrm{u}[n]=\frac{1}{2^n}\left[\cos\left(\frac{\pi}{2}n\right)+\sin\left(\frac{\pi}{2}n\right)\right]\mathrm{u}[n]$$

$\mathrm{h}[0]=\ 1$

$\mathrm{h}[1]=\ \frac{1}{2}$

$\mathrm{h}[2]=\ -\frac{1}{4}$ Check.

2. Take the unilateral Laplace transform of both sides of the following differential equation, apply the stated initial conditions and find an exact numerical solution, in terms of real-valued functions only, for time, $t>0$.

$$\frac{d^2}{dt^2}(\mathrm{y}(t))+7\frac{d}{dt}(\mathrm{y}(t))+12\,\mathrm{y}(t)=\delta(t-1)$$

$$\mathrm{y}(0^+)=1\ ,\ \left.\frac{d}{dt}(\mathrm{y}(t))\right|_{t=0^+}=-1$$

$$s^2\,\mathrm{Y}(s)-s\,\mathrm{y}(0^+)-\left[\frac{d}{dt}(\mathrm{y}(t))\right]_{t=0^+}+7\left[s\,\mathrm{Y}(s)-\mathrm{y}(0^+)\right]+12\,\mathrm{Y}(s)=e^{-s}$$

$$s^2\,\mathrm{Y}(s)+7s\,\mathrm{Y}(s)+12\,\mathrm{Y}(s)=s+6+e^{-s}$$

$$\mathrm{Y}(s) = \frac{s+6}{s^2+7s+12} + \frac{1}{s^2+7s+12}e^{-s} = \frac{s+6}{(s+3)(s+4)} + \frac{1}{(s+3)(s+4)}e^{-s}$$

$$\mathrm{Y}(s) = \frac{3}{s+3} - \frac{2}{s+4} + \left(\frac{1}{s+3} - \frac{1}{s+4}\right)e^{-s}$$

$$\mathrm{y}(t) = \left[3e^{-3t} - 2e^{-4t}\right]\mathrm{u}(t) + \left[e^{-3(t-1)} - e^{-4(t-1)}\right]\mathrm{u}(t-1)$$

3. The signal, $\mathrm{x}(t) = 10\sin(2\pi t)\cos(5\pi t)$, is sampled over exactly one period, with the first sample taken at time, $t = 0$. The sampling rate is the smallest integer multiple of the fundamental frequency of the signal that is *greater than* (not "greater than or equal to") the Nyquist rate of the signal.

(a) What is the period of $\mathrm{x}(t)$?

Using $\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$

$$\mathrm{x}(t) = 5\left[\sin(-3\pi t) + \sin(7\pi t)\right] = 5\left[\sin(7\pi t) - \sin(3\pi t)\right]$$

Fundamental frequency is the greatest common divisor of 3.5 Hz and 1.5 Hz or 0.5 Hz. Therefore the period is 2 s.

(b) How many samples are taken?

The Nyquist frequency is 3.5 Hz. Therefore the Nyquist rate is 7 Hz which is 14 times the fundamental frequency. The smallest integer multiple of 0.5 Hz that is greater than 7 Hz is 7.5 Hz which yields 15 samples in 2 seconds.

(c) What are the numerical values of the first three samples?

The times of the first three samples are $t = 0$, $t = \frac{2}{15}$ and $t = \frac{4}{15}$.

$$\mathrm{x}(0) = 10\sin(2\pi t)\cos(5\pi t) = 0$$

$$\mathrm{x}\left(\frac{2}{15}\right) = 10\sin\left(\frac{4\pi}{15}\right)\cos\left(\frac{10\pi}{15}\right) = -3.715$$

$$\mathrm{x}\left(\frac{4}{15}\right) = 10\sin\left(\frac{8\pi}{15}\right)\cos\left(\frac{20\pi}{15}\right) = -4.972$$

4. (a) A complex variable, *w*, is related to a complex variable, *z*, through the relationship,

$$w = \frac{1}{z}$$

where $z = x + jy = re^{j\theta}$ and $w = u + jv = \rho e^{j\phi}$ Describe mathematically and sketch in the [w] plane all possible *w* regions, corresponding to the *z* region, $1 < r < 2$, $0 < \theta < \frac{\pi}{2}$.

$$\rho e^{j\phi} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta}$$

Therefore the region, $1 < r < 2$, $0 < \theta < \frac{\pi}{2}$, maps into the region, $\frac{1}{2} < \rho < 1$, $0 > \phi > -\frac{\pi}{2}$

(b) A complex variable, *w*, is related to a complex variable, *z*, through the relationship,

$$w = \frac{1}{z^2}$$

where $z = x + jy = re^{j\theta}$ and $w = u + jv = \rho e^{j\phi}$ Describe mathematically and sketch in the [z] plane all possible "z" regions, corresponding to the "w" region, $1 < \rho < 3$, $\frac{\pi}{2} < \phi < \pi$.

$$z = re^{j\theta} = \pm\frac{1}{\sqrt{w}} = \pm\frac{1}{\sqrt{\rho e^{j\phi}}} = \pm\frac{1}{\sqrt{\rho}}e^{-j\frac{\phi}{2}} = \frac{1}{\sqrt{\rho}}e^{-j\frac{\phi}{2}} \text{ or } \frac{1}{\sqrt{\rho}}e^{-j\left(\frac{\phi}{2}-\pi\right)}$$

Therefore the region, $1 < \rho < 3$, $\frac{\pi}{2} < \phi < \pi$, maps into the two regions, $1 < r < \sqrt{3}$, $-\frac{\pi}{4} > \theta > -\frac{\pi}{2}$ and $1 < r < \sqrt{3}$, $\frac{3\pi}{4} > \theta > \frac{\pi}{2}$.

5. A system is described by the differential equation,

$$\frac{d^2}{dt^2}[\mathrm{y}(t)] + 5\frac{d}{dt}[\mathrm{y}(t)] + 6\,\mathrm{y}(t) = \mathrm{u}(t) \quad .$$

The initial conditions of the system at time, $t = 0^+$, are

$$\mathrm{y}(0^+) = 1 \text{ and } \frac{d}{dt}[\mathrm{y}(t)]\bigg|_{t=0^+} = -2$$

Laplace transform both sides of the equation and solve for the Laplace transform of y(*t*), Y(*s*). Then find y(*t*) by taking the inverse Laplace transform of that result. . Check your solution by substituting it into the differential equation and also check to be sure that your solution satisfies both initial conditions.

$$s^2\,\mathrm{Y}(s) - sy(0^+) - \frac{d}{dt}[\mathrm{y}(t)]\bigg|_{t=0^+} + 5[s\,\mathrm{Y}(s) - y(0^+)] + 6\,\mathrm{Y}(s) = \frac{1}{s}$$

$$s^2\,\mathrm{Y}(s) - s + 2 + 5s\,\mathrm{Y}(s) - 5 + 6\,\mathrm{Y}(s) = \frac{1}{s}$$

$$\mathrm{Y}(s)=\frac{\frac{1}{s}+s+3}{s^2+5s+6}$$

$$\mathrm{Y}(s)=\frac{s^2+3s+1}{s\left(s^2+5s+6\right)}=\frac{s^2+3s+1}{s(s+2)(s+3)}$$

$$\mathrm{Y}(s)=\frac{\frac{1}{6}}{s}+\frac{\frac{1}{2}}{s+2}+\frac{\frac{1}{3}}{s+3}$$

$$\mathrm{y}(t)=\left[\frac{1}{6}+\frac{1}{2}e^{-2t}+\frac{1}{3}e^{-3t}\right]\mathrm{u}(t)$$

6. Find the Nyquist rate for these signals. (The Nyquist rate is the minimum sampling rate required to avoid aliasing, as determined by Shannon's sampling theorem.)

(a) $\mathrm{x}(t)=10\,\mathrm{sinc}(200t)$ $\mathrm{X}(f)=\frac{1}{20}\mathrm{rect}\left(\frac{f}{200}\right)\Rightarrow f_s=200$

(b) $\mathrm{x}(t)=\sin(20\pi t)\cos(2000\pi t)$

$$\mathrm{X}(f)=\frac{j}{2}\left[\delta(f+10)-\delta(f-10)\right]*\frac{1}{2}\left[\delta(f-1000)+\delta(f+1000)\right]$$

$$\mathrm{X}(f)=\frac{j}{4}\left[\delta(f-990)-\delta(f+1010)-\delta(f-1010)-\delta(f+990)\right]$$

$$f_s=2020$$

7. Using the impulse invariant digital filter design technique, with a sampling rate of 10 Hz, find the z-transform transfer function, $\mathrm{H}_z(z)$, corresponding to

$$\mathrm{H}_s(s)=10\frac{s+3}{s^2+7s+6}\quad .$$

Then write a recursion relation, with real-valued numerical coefficients, relating the response of the system, $\mathrm{y}[n]$, to the present and previous excitations to the system, $\mathrm{x}[n]$, $\mathrm{x}[n-1]$, $\mathrm{x}[n-2]$, $\cdots$ and the previous responses of the system, $\mathrm{y}[n-1]$, $\mathrm{y}[n-2]$, $\cdots$.

$$\mathrm{H}_s(s) = 10\frac{s+3}{(s+1)(s+6)} = 10\left(\frac{\frac{2}{5}}{s+1} + \frac{\frac{3}{5}}{s+6}\right) = 2\left(\frac{2}{s+1} + \frac{3}{s+6}\right)$$

$$\mathrm{h}(t) = 2\left(2e^{-t} + 3e^{-6t}\right)\mathrm{u}(t)$$

Using

$$Ke^{-at}\,\mathrm{u}(t) \leftrightarrow \frac{Kz}{z - e^{-aT_s}}$$

$$\mathrm{H}_z(z) = 2\left(\frac{2z}{z-e^{-T_s}} + \frac{3z}{z-e^{-6T_s}}\right)$$

$$\mathrm{H}_z(z) = 2\frac{2z\left(z-e^{-6T_s}\right) + 3z\left(z-e^{-T_s}\right)}{\left(z-e^{-T_s}\right)\left(z-e^{-6T_s}\right)} = 2z\frac{5z - 2e^{-6T_s} - 3e^{-T_s}}{z^2 - z\left(e^{-T_s} + e^{-6T_s}\right) + e^{-7T_s}}$$

$$\mathrm{H}_z(z) = 2\frac{2z\left(z-e^{-6T_s}\right) + 3z\left(z-e^{-T_s}\right)}{\left(z-e^{-T_s}\right)\left(z-e^{-6T_s}\right)} = 2z\frac{5z - 2e^{-6T_s} - 3e^{-T_s}}{z^2 - z\left(e^{-T_s} + e^{-6T_s}\right) + e^{-7T_s}}$$

$$\frac{\mathrm{Y}_z(z)}{\mathrm{X}_z(z)} = 2z\frac{5z - 2e^{-6T_s} - 3e^{-T_s}}{z^2 - z\left(e^{-T_s} + e^{-6T_s}\right) + e^{-7T_s}} = 2z\frac{5z - 2e^{-0.6} - 3e^{-0.1}}{z^2 - z\left(e^{-0.1} + e^{-0.6}\right) + e^{-0.7}}$$

$$\frac{\mathrm{Y}_z(z)}{\mathrm{X}_z(z)} = 2z\frac{5z - 2(0.549) - 3(0.905)}{z^2 - z(0.549 + 0.905) + 0.497} = 2z\frac{5z - 3.813}{z^2 - 1.454z + 0.497}$$

$$\mathrm{y}[n] = 10\,\mathrm{x}[n] - 7.626\,\mathrm{x}[n-1] + 1.454\,\mathrm{y}[n-1] - 0.497\,\mathrm{y}[n-2]$$

8. Find all the distinct roots of

$$z^5 = -1 \quad .$$

The "nth" root, z_0, of a complex number, z, is the solution to the equation,

$$z_0^n = z$$

where n is a positive integer. In polar form,

$$z = r\left[\cos(\theta) + j\sin(\theta)\right] \text{ and } z_0 = r_0\left[\cos(\theta_0) + j\sin(\theta_0)\right] \quad .$$

Then

$$r_0^n\left[\cos(n\theta_0) + j\sin(n\theta_0)\right] = r\left[\cos(\theta) + j\sin(\theta)\right]$$

and the solution for r_0 and θ_0 is

$$r_0 = \sqrt[n]{r} \quad \text{and} \quad \theta_0 = \frac{\theta \pm 2k\pi}{n} \quad , \; k \text{ an integer,}$$

and there are exactly *n* distinct values. In this case,

$$r = 1 \quad \text{and} \quad \theta = \pi$$

Therefore

$$r_0 = \sqrt[n]{1} \quad \text{and} \quad \theta_0 = \frac{\pi \pm 2k\pi}{5} \quad , \; k \text{ an integer,}$$

Therefore the five distinct roots are

$$e^{j\frac{\pi}{5}} , e^{j\frac{3\pi}{5}} , e^{j\pi} , e^{j\frac{7\pi}{5}} , e^{j\frac{9\pi}{5}}$$

or, in rectangular form,

$$0.809 + j0.588 , -0.309 + j0.951 , -1 , -0.309 - j0.951 , 0.809 - j0.588 \quad .$$

9. Evaluate the following definite integral using contour integration in the complex plane and the method of residues.

$$I = \int_0^\infty \frac{dx}{x^4 + 1}$$

Since the integrand is an even function,

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dz}{z^4 + 1} \quad .$$

Letting the path of integration be part of a closed path which closes in a counter-clockwise direction with an infinite radius in the upper half-plane,

$$I = \frac{1}{2} j2\pi \sum \text{residues} \quad .$$

The integrand can be factored into

$$\frac{1}{z^4 + 1} = \frac{1}{\left(z - \frac{1+j}{\sqrt{2}}\right)\left(z - \frac{-1+j}{\sqrt{2}}\right)\left(z - \frac{-1-j}{\sqrt{2}}\right)\left(z - \frac{1-j}{\sqrt{2}}\right)} \quad .$$

Then

$$I = j\pi\left(\text{Residue at } z = \frac{1+j}{\sqrt{2}} + \text{Residue at } z = \frac{-1+j}{\sqrt{2}}\right)$$

$$= j\pi\left[\begin{array}{l}\dfrac{1}{\left(\frac{1+j}{\sqrt{2}} - \frac{-1+j}{\sqrt{2}}\right)\left(\frac{1+j}{\sqrt{2}} - \frac{-1-j}{\sqrt{2}}\right)\left(\frac{1+j}{\sqrt{2}} - \frac{1-j}{\sqrt{2}}\right)} \\ + \dfrac{1}{\left(\frac{-1+j}{\sqrt{2}} - \frac{1+j}{\sqrt{2}}\right)\left(\frac{-1+j}{\sqrt{2}} - \frac{-1-j}{\sqrt{2}}\right)\left(\frac{-1+j}{\sqrt{2}} - \frac{1-j}{\sqrt{2}}\right)}\end{array}\right]$$

$$I = j\pi\left[\frac{1}{\sqrt{2}\sqrt{2}(1+j)j\sqrt{2}} + \frac{1}{\left(-\sqrt{2}\right)\left(j\sqrt{2}\right)\sqrt{2}(-1+j)}\right]$$

$$I = \pi\left[\frac{1}{2\sqrt{2}(1+j)} - \frac{1}{2\sqrt{2}(-1+j)}\right] = \frac{\pi}{2\sqrt{2}}\left[\frac{1}{1+j} + \frac{1}{1-j}\right]$$

$$I = \frac{\pi}{2\sqrt{2}}$$

10. Find the inverse Fourier transform, $\mathrm{x}(t)$, of

$$\mathrm{X}(f) = 10\left[\mathrm{rect}\left(\frac{f-4}{2}\right) + \mathrm{rect}\left(\frac{f+4}{2}\right)\right]e^{-j2\pi f} \quad .$$

Express $\mathrm{x}(t)$ in as simply as possible in terms of real functions of time.

$$\mathrm{sinc}(t) \xleftrightarrow{F} \mathrm{rect}(f)$$

$$2\,\mathrm{sinc}(2t) \xleftrightarrow{F} \mathrm{rect}\left(\frac{f}{2}\right)$$

$$2\,\mathrm{sinc}(2t)e^{j8\pi t} \xleftrightarrow{F} \mathrm{rect}\left(\frac{f-4}{2}\right)$$

$$2\,\mathrm{sinc}(2t)e^{-j8\pi t} \xleftrightarrow{F} \mathrm{rect}\left(\frac{f+4}{2}\right)$$

$$2\,\mathrm{sinc}(2t)\left[e^{j8\pi t} + e^{-j8\pi t}\right] \xleftrightarrow{F} \mathrm{rect}\left(\frac{f-4}{2}\right) + \mathrm{rect}\left(\frac{f+4}{2}\right)$$

$$2\,\mathrm{sinc}(2t)\left[2\cos(8\pi t)\right] \xleftrightarrow{F} \mathrm{rect}\left(\frac{f-4}{2}\right) + \mathrm{rect}\left(\frac{f+4}{2}\right)$$

$$4\,\text{sinc}(2t)\cos(8\pi t) \xleftrightarrow{F} \text{rect}\left(\frac{f-4}{2}\right) + \text{rect}\left(\frac{f+4}{2}\right)$$

$$40\,\text{sinc}(2t)\cos(8\pi t) \xleftrightarrow{F} 10\left[\text{rect}\left(\frac{f-4}{2}\right) + \text{rect}\left(\frac{f+4}{2}\right)\right]$$

$$40\,\text{sinc}[2(t-1)]\cos[8\pi(t-1)] \xleftrightarrow{F} 10\left[\text{rect}\left(\frac{f-4}{2}\right) + \text{rect}\left(\frac{f+4}{2}\right)\right]e^{-j2\pi f}$$

Therefore

$$x(t) = 40\,\text{sinc}[2(t-1)]\cos[8\pi(t-1)]$$

11. Four samples, {x}, are taken at the Nyquist rate from one period of a bandlimited, periodic signal and their discrete Fourier transform, {X}, is found to be

$$\{X\} = \{2, 1-j, 4, 1+j\} \quad .$$

(a) What is the average value of the original bandlimited periodic signal?

The average value of the original signal is the $X[0]$ in the complex Fourier series.

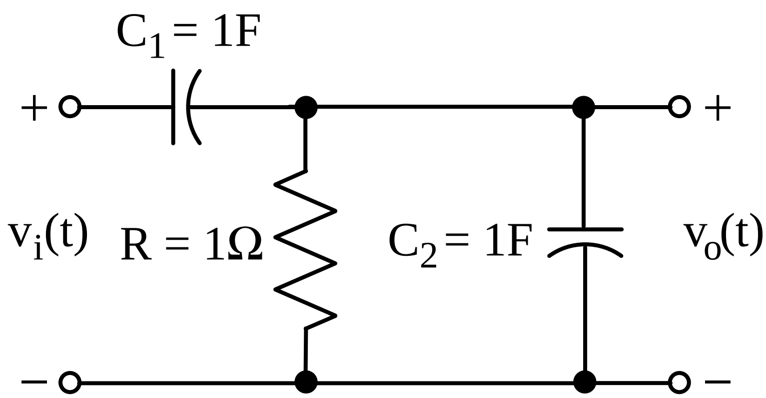
$$X_{CTFS}[0] = \frac{X_{DFT}[0]}{N} = \frac{2}{4} = \frac{1}{2}$$

(b) What was the first sample value, $x[0]$?

$$x[n] = \frac{1}{N_F}\sum_{k=0}^{N_F-1} X[k] e^{j\frac{2\pi nk}{N_F}}$$

$$x[0] = \frac{1}{4}\sum_{k=0}^{3} X[k] = \frac{1}{4}(2 + 1 - j + 4 + 1 + j) = \frac{8}{4} = 2$$

12. If the excitation voltage, $v_i(t)$, to the circuit below is a unit step, find a real, time-domain expression for the response voltage, $v_o(t)$. (Assume the capacitors are initially uncharged.)



The Laplace-domain transfer function is

$$\mathrm{H}(s)=\frac{\dfrac{\dfrac{R}{sC_2}}{\dfrac{1}{sC_2}+R}}{\dfrac{\dfrac{R}{sC_2}}{\dfrac{1}{sC_2}+R}+\dfrac{1}{sC_1}}=\frac{sRC_1}{1+s(C_1+C_2)R}$$

$$\mathrm{V}_o(s)=\frac{1}{s}\frac{sRC_1}{1+s(C_1+C_2)R}=\frac{RC_1}{1+s(C_1+C_2)R}=\frac{\dfrac{C_1}{C_1+C_2}}{s+\dfrac{1}{R(C_1+C_2)}}$$

Then, using

$$e^{-\alpha t}\xleftrightarrow{F}\frac{1}{s+\alpha}$$

$$\mathrm{v}_o(t)=\frac{C_1}{C_1+C_2}e^{-\frac{t}{R(C_1+C_2)}}\mathrm{u}(t)$$

$$\mathrm{v}_o(t)=\frac{1}{2}e^{-\frac{t}{2}}\mathrm{u}(t)$$

13. An analog filter has a transfer function,

$$\mathrm{H}_s(s)=\frac{25}{s^2+8s+52}\quad .$$

Using a time between samples of 10 ms, find the corresponding $H_z(z)$ using the impulse-invariant digital filter synthesis technique.

$$H_s(s) = \frac{25}{(s+4-j6)(s+4+j6)} = 25\left(\frac{\frac{1}{j12}}{s+4-j6} - \frac{\frac{1}{j12}}{s+4+j6}\right)$$

Then, using

$$e^{-\alpha t} \xleftrightarrow{F} \frac{1}{s+\alpha}$$

$$h(t) = -j\frac{25}{12}\left[e^{-(4-j6)t} - e^{-(4+j6)t}\right]u(t) = \frac{50}{12}\frac{e^{-(4-j6)t} - e^{-(4+j6)t}}{j2}u(t) = \frac{25}{6}e^{-4t}\sin(6t)u(t)$$

Then, using

$$Ke^{-at} \xleftrightarrow{F} \frac{Kz}{z-e^{-aT_s}}$$

$$H_z(z) = -j\frac{25}{12}\left[\frac{z}{z-e^{-(4-j6)T_s}} - \frac{z}{z-e^{-(4+j6)T_s}}\right]$$

$$H_z(z) = -j\frac{25}{12}\frac{z\left[z-e^{-(4+j6)T_s}\right] - z\left[z-e^{-(4-j6)T_s}\right]}{\left[z-e^{-(4-j6)T_s}\right]\left[z-e^{-(4+j6)T_s}\right]}$$

$$H_z(z) = -j\frac{25}{12}\frac{ze^{-(4-j6)T_s} - ze^{-(4+j6)T_s}}{z^2 - z\left[e^{-(4-j6)T_s} + e^{-(4+j6)T_s}\right] + e^{-8T_s}}$$

$$H_z(z) = -j\frac{25}{12}\frac{ze^{-4T_s}\left(e^{j6T_s} - e^{-j6T_s}\right)}{z^2 - ze^{-4T_s}\left(e^{j6T_s} + e^{-j6T_s}\right) + e^{-8T_s}}$$

$$H_z(z) = \frac{25}{12}\frac{ze^{-4T_s}2\sin(6T_s)}{z^2 - ze^{-4T_s}2\cos(6T_s) + e^{-8T_s}}$$

$$H_z(z) = \frac{25}{6}\frac{ze^{-0.04}\sin(0.06)}{z^2 - ze^{-0.04}2\cos(0.06) + e^{-0.08}} = \frac{0.24z}{z^2 - 1.918z + 0.923}$$

14. A sampled-data system operates at a sampling rate of 1 kHz with a z-domain transfer function of

$$H_z(z) = \frac{1-0.4z^{-1}}{1-1.4z^{-1}+0.45z^{-2}} .$$

(a) Find and list all the poles and zeros of this transfer function.

$$\mathrm{H}_z(z) = \frac{1-0.4z^{-1}}{1-1.4z^{-1}+0.45z^{-2}} = \frac{1-0.4z^{-1}}{\left(1-0.5z^{-1}\right)\left(1-0.9z^{-1}\right)} = z\frac{z-0.4}{(z-0.5)(z-0.9)}$$

Zeros at $z=0$ and $z=0.4$.

Poles at $z=0.5$ and $z=0.9$.

(b) Write the recursion relation between the excitation, x, and the response, y, in the form,

$$\mathrm{y}[n] = a_n\,\mathrm{x}[n] + a_n\,\mathrm{x}[n-1] + \cdots + b_n\,\mathrm{y}[n-1] + b_n\,\mathrm{y}[n-2] + \cdots \ .$$

$$\mathrm{y}[n] = \mathrm{x}[n] - 0.4\,\mathrm{x}[n-1] + 1.4\,\mathrm{y}[n-1] - 0.45\,\mathrm{y}[n-2]$$

(c) If the excitation x consists of an infinite sequence of samples beginning at time $t=0$, all of which are the value, "1", what value does the response, $\mathrm{y}[n]$ approach as n approaches infinity?

As y approaches a steady value, all the delayed versions of y are the same. The x values are already all the same, 1. Therefore in that limit,

$$\mathrm{y}[n] = 1 - 0.4 + 1.4\,\mathrm{y}[n] - 0.45\,\mathrm{y}[n]$$

or

$$0.05\,\mathrm{y}[n] = 0.6$$

or

$$\mathrm{y}[n] = \frac{0.6}{0.05} = 12$$

15. Use the Laplace transform to solve this differential equation with initial conditions:

$$\frac{d^2}{dt^2}[\mathrm{x}(t)] + 8\frac{d}{dt}[\mathrm{x}(t)] + 7\,\mathrm{x}(t) = \mathrm{u}(t) \ \ , \ \ x\left(0^+\right) = -2 \ \ , \ \ \frac{d}{dt}[x(t)]_{t=0^+} = 2$$

Show that your solution yields the correct initial conditions.

Laplace transforming,

$$s^2\,\mathrm{X}(s) - s\,\mathrm{x}\left(0^+\right) - \frac{d}{dt}[\mathrm{x}(t)]_{t=0^+} + 8\left[s\,\mathrm{X}(s) - \mathrm{x}\left(0^+\right)\right] + 7\,\mathrm{X}(s) = \frac{1}{s}$$

Solving for $\mathrm{X}(s)$,

$$s^2\,\mathrm{X}(s) + 8s\,\mathrm{X}(s) + 7\,\mathrm{X}(s) = \frac{1}{s} + s\,\mathrm{x}\left(0^+\right) + \frac{d}{dt}[\mathrm{x}(t)]_{t=0^+} + 8\,\mathrm{x}\left(0^+\right)$$

or

$$\left(s^2 + 8s + 7\right)\mathrm{X}(s) = \frac{1}{s} - 2s - 14$$

or

$$\mathrm{X}(s) = \frac{\frac{1}{s} - 2s - 14}{s^2 + 8s + 7} = \frac{1 - 2s^2 - 14s}{s(s^2 + 8s + 7)} = -\frac{2s^2 + 14s - 1}{s(s+1)(s+7)}$$

Expanding in partial fractions,

$$\mathrm{X}(s) = \frac{\frac{1}{7}}{s} - \frac{\frac{13}{6}}{s+1} + \frac{\frac{1}{42}}{s+7}$$

Inverse Laplace transforming,

$$\mathrm{x}(t) = \left(\frac{1}{7} - \frac{13}{6}e^{-t} + \frac{1}{42}e^{-7t}\right)\mathrm{u}(t)$$

Verifying initial conditions,

$$\mathrm{x}(0^+) = \frac{1}{7} - \frac{13}{6} + \frac{1}{42} = \frac{6 - 91 + 1}{42} = -2 \quad \text{Check.}$$

$$\frac{d}{dt}[\mathrm{x}(t)] = \frac{1}{7}\delta(t) - \left[\frac{13}{6}e^{-t}\delta(t) - \frac{13}{6}e^{-t}\,\mathrm{u}(t)\right] + \left[\frac{1}{42}e^{-7t}\delta(t) - \frac{1}{6}e^{-7t}\,\mathrm{u}(t)\right]$$

$$\frac{d}{dt}[\mathrm{x}(t)]_{t=0^+} = \frac{13}{6} - \frac{1}{6} = 2 \quad \text{Check.}$$

16. For each two-dimensional space-domain function, $\mathrm{g}(x,y)$, below find its two-dimensional Fourier transform, $\mathrm{G}(f_x, f_y)$, and then find the numerical value of its two-dimensional Fourier transform at the spatial frequencies, f_x and f_y, given.

(a) $\mathrm{g}(x,y) = 10\,\mathrm{rect}(5x)\,\mathrm{sinc}\left(\frac{y}{2}\right)$

$$\mathrm{G}(f_x, f_y) = \frac{10}{5}\mathrm{sinc}\left(\frac{f_x}{5}\right)2\,\mathrm{rect}(2f_y) = 4\,\mathrm{sinc}\left(\frac{f_x}{5}\right)\mathrm{rect}(2f_y)$$

$$\mathrm{G}(1,0) = 4\,\mathrm{sinc}\left(\frac{1}{5}\right) = 3.742$$

$\mathrm{G}(1,0) = 3.742$

(b) $\mathrm{g}(x,y) = 4\,\mathrm{sinc}\left(\frac{x}{4}\right)\mathrm{comb}(2x)\,\mathrm{tri}(6y)$

$$\mathrm{G}(f_x, f_y) = \left[16\,\mathrm{rect}(4f_x) * \frac{1}{2}\mathrm{comb}\left(\frac{f_x}{2}\right)\right]\frac{1}{6}\mathrm{sinc}^2\left(\frac{f_y}{6}\right)$$

$$\mathrm{G}(f_x, f_y) = \frac{4}{3}\left[\mathrm{rect}(4f_x) * \sum_{n=-\infty}^{\infty}\delta\left(\frac{f_x}{2} - n\right)\right]\mathrm{sinc}^2\left(\frac{f_y}{6}\right)$$

$$\mathrm{G}(f_x, f_y) = \frac{8}{3}\left[\sum_{n=-\infty}^{\infty} \mathrm{rect}(4f_x) * \delta(f_x - 2n)\right]\mathrm{sinc}^2\left(\frac{f_y}{6}\right)$$

$$\mathrm{G}(f_x, f_y) = \frac{8}{3}\left\{\sum_{n=-\infty}^{\infty} \mathrm{rect}[4(f_x - 2n)]\right\}\mathrm{sinc}^2\left(\frac{f_y}{6}\right)$$

$$\mathrm{G}(0,3) = \frac{8}{3}\left\{\underbrace{\sum_{n=-\infty}^{\infty} \mathrm{rect}(-8n)}_{=1}\right\}\mathrm{sinc}^2\left(\frac{1}{2}\right) = \frac{8}{3}\mathrm{sinc}^2\left(\frac{1}{2}\right) = \frac{8}{3}\left(\frac{2}{\pi}\right)^2 = \frac{32}{3\pi^2} = 1.0808$$

$\mathrm{G}(0,3) = 1.080$

17. A function, $\mathrm{y}(t)$, is given by

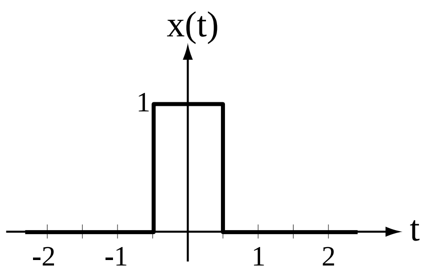
$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) \quad .$$

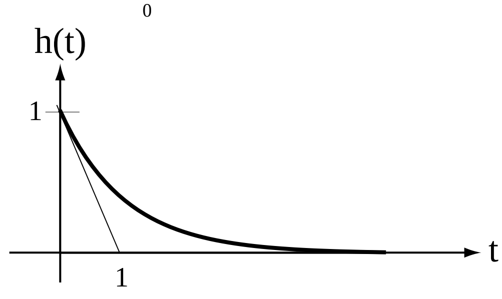
Let $\mathrm{x}(t) = \mathrm{rect}(t)$ and let $\mathrm{h}(t) = e^{-t}\,\mathrm{u}(t)$. Using the definition of convolution,

$$\mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)d\tau = \int_{-\infty}^{\infty} \mathrm{h}(\tau)\mathrm{x}(t-\tau)d\tau \,,$$

convolve x with h to find an expression for $\mathrm{y}(t)$ and then sketch $\mathrm{y}(t)$.

$$\mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} e^{-\tau}\,\mathrm{u}(\tau)\mathrm{rect}(t-\tau)d\tau = \int_{0}^{\infty} e^{-\tau}\,\mathrm{rect}(t-\tau)d\tau$$





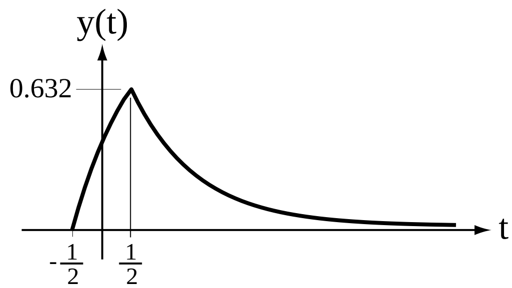
From the graph, for $t < -\frac{1}{2}$ the integral is zero.

For $-\frac{1}{2} < t < \frac{1}{2}$:

$$\mathrm{x}(t) * \mathrm{h}(t) = \int_{0}^{t+\frac{1}{2}} e^{-\tau}\,d\tau = \left[-e^{-\tau}\right]_0^{t+\frac{1}{2}} = 1 - e^{-\left(t+\frac{1}{2}\right)}$$

For $t > \frac{1}{2}$:

$$x(t) * h(t) = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} e^{-\tau}\, d\tau = \left[-e^{-\tau}\right]_{t-\frac{1}{2}}^{t+\frac{1}{2}} = e^{-\left(t-\frac{1}{2}\right)} - e^{-\left(t+\frac{1}{2}\right)}$$



18. An analog filter has a transfer function,

$$H(s) = \frac{10}{(s+1)(s+2)} \quad .$$

(a) Find a real-valued functional expression for its step response, $h^{-1}(t)$.

$$h^{-1}(t) = L^{-1}\left[\frac{H(s)}{s}\right] = L^{-1}\left[\frac{10}{s(s+1)(s+2)}\right] = L^{-1}\left[\frac{5}{s} - \frac{10}{s+1} + \frac{5}{s+2}\right]$$

$$h^{-1}(t) = 5\left[1 - 2e^{-t} + 1e^{-2t}\right]u(t)$$

(b) Find a recursion relation for a digital filter operating with a sampling time, $T_S = 0.1$, which accepts the infinite-length excitation sequence,

$$\{1,1,1,1,1,\ldots\} ,$$

and produces an response sequence which consists of samples from the step response of the analog filter. That is, the response sequence produced by the digital filter should be

$$\left\{h^{-1}(0)\, ,\, h^{-1}(T_s)\, ,\, h^{-1}(2T_s)\, ,\ldots\right\} .$$

If the transfer function of the analog filter is

$$H_s(s)$$

then the step response is

$$h^{-1}(t) = L^{-1}\left[\frac{H_s(s)}{s}\right] \quad .$$

The z transform of this step response is

$$Z\left[\mathrm{h}^{-1}(t)\right]$$

which must equal the *z*-domain step response form

$$\frac{z}{z-1}\mathrm{H}_z(z)$$

where $\mathrm{H}_z(z)$ is the z domain transfer function. That is,

$$\mathrm{H}_z(z) = \frac{z-1}{z} Z\left\{L^{-1}\left[\frac{\mathrm{H}_s(s)}{s}\right]\right\} = \frac{z-1}{z} Z\left[\mathrm{h}^{-1}(t)\right] \quad .$$

$$\mathrm{H}_z(z) = \frac{z-1}{z} Z\left\{5\left[1-2e^{-t}+1e^{-2t}\right]\mathrm{u}(t)\right\}$$

$$\mathrm{H}_z(z) = \frac{z-1}{z} Z\left\{5\left[1-2e^{-t}+1e^{-2t}\right]\mathrm{u}(t)\right\}$$

Using $K\,\mathrm{u}(t) \xleftrightarrow{F} \dfrac{Kz}{z-1}$ and $Ke^{-at}\,\mathrm{u}(t) \xleftrightarrow{F} \dfrac{Kz}{z-e^{-aT_s}}$,

$$\mathrm{H}_z(z) = \frac{z-1}{z} 5\left[\frac{z}{z-1} - \frac{2z}{z-e^{-T_s}} + \frac{z}{z-e^{-2T_s}}\right]$$

$$\frac{\mathrm{Y}_z(z)}{\mathrm{X}_z(z)} = \mathrm{H}_z(z) = 5\frac{z\left[1-2e^{-T_s}+e^{-2T_s}\right]+e^{-T_s}-2e^{-2T_s}+e^{-3T_s}}{z^2 - z\left(e^{-T_s}+e^{-2T_s}\right)+e^{-3T_s}}$$

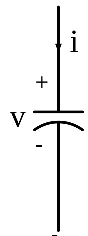
$$\mathrm{Y}_z(z)\left[z^2 - z\left(e^{-T_s}+e^{-2T_s}\right)+e^{-3T_s}\right] = 5\left\{z\left[1-2e^{-T_s}+e^{-2T_s}\right]+e^{-T_s}-2e^{-2T_s}+e^{-3T_s}\right\}\mathrm{X}_z(z)$$

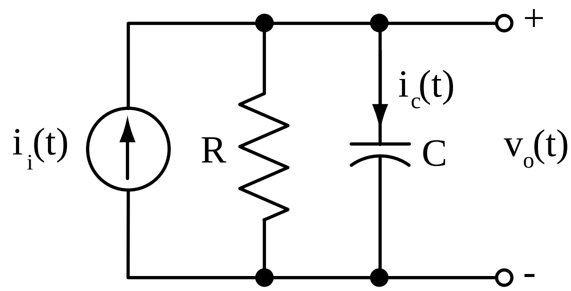
$$\mathrm{Y}_z(z)\left[1 - z^{-1}\left(e^{-T_s}+e^{-2T_s}\right)+z^{-2}e^{-3T_s}\right] = 5\left\{z^{-1}\left[1-2e^{-T_s}+e^{-2T_s}\right]+z^{-2}\left(e^{-T_s}-2e^{-2T_s}+e^{-3T_s}\right)\right\}\mathrm{X}_z(z)$$

$$\mathrm{y}[n] = 5\left[1-2e^{-T_s}+e^{-2T_s}\right]\mathrm{x}[n-1] + 5\left(e^{-T_s}-2e^{-2T_s}+e^{-3T_s}\right)\mathrm{x}[n-2] + \left(e^{-T_s}+e^{-2T_s}\right)\mathrm{y}[n-1] - e^{-3T_s}\,\mathrm{y}[n-2]$$

$$\mathrm{y}[n] = 0.3145\,\mathrm{x}[n-1] + 0.041\,\mathrm{x}[n-2] + 1.7235\,\mathrm{y}[n-1] - 0.7408\,\mathrm{y}[n-2]$$

19. Write a differential equation for the sum of currents at one node in the circuit below equal to zero. Then, using the Laplace transform, solve the differential equation for the real, time-domain expression for the voltage, $\mathrm{v}_o(t)$, if the excitation current, $\mathrm{i}_i(t)$ is a unit step and the initial condition is $\mathrm{v}_o(0^+) = -1$. Let the component values be $R = 1\ \Omega$ and $C = 1\ \mathrm{F}$.





$$i = C\frac{d}{dt}[v(t)]$$

The differential equation describing the circuit is

$$\mathrm{i}_i(t) = \mathrm{u}(t) = \frac{\mathrm{v}_o(t)}{R} + \mathrm{i}_c(t) \quad .$$

Laplace transforming, using $\frac{d}{dt}\mathrm{g}(t) \xleftrightarrow{F} s\mathrm{G}(s) - \mathrm{g}(0^+)$,

$$\mathrm{I}_i(s) = \frac{1}{s} = \frac{\mathrm{V}_o(s)}{R} + sC\,\mathrm{V}_o(s) - C\,\mathrm{v}_o(0^+)$$

$$\mathrm{V}_o(s) = \frac{\frac{1}{s} + C\,\mathrm{v}_o(0^+)}{\frac{1}{R} + sC} = R\frac{sC\,\mathrm{v}_o(0^+) + 1}{s(sRC + 1)} = \frac{1}{C}\frac{sC\,\mathrm{v}_o(0^+) + 1}{s\left(s + \frac{1}{RC}\right)}$$

$$\mathrm{V}_o(s) = \frac{R}{s} + \frac{\mathrm{v}_o(0^+) - R}{s + \frac{1}{RC}}$$

$$\mathrm{v}_o(t) = \left\{R + \left[\mathrm{v}_o(0^+) - R\right]e^{-\frac{t}{RC}}\right\}\mathrm{u}(t) = \left(1 - 2e^{-t}\right)\mathrm{u}(t)$$

20. A real-valued, bandlimited, periodic signal, $\mathrm{x}(t)$, is sampled over exactly one period at exactly its Nyquist rate of 1 kHz. The set of samples is {x} and there are 8 total samples in {x}. Some of the data returned by the discrete Fourier transform (DFT) of {x} are

$$\mathrm{X}[0] = 4 \text{ , } \mathrm{X}[1] = 4 + j2 \text{ , } \mathrm{X}[2] = -6 + j4 \text{ , } \mathrm{X}[3] = 4 - j10 \text{ , } \mathrm{X}[4] = -2 \quad .$$

(a) What is the numerical value of $\mathrm{X}[5]$?

$$\mathrm{X}[5] = \mathrm{X}^*[3] = 4 + j10$$

(b) A complex Fourier series describing $\mathrm{x}(t)$ is found using one period of $\mathrm{x}(t)$ as the time, T_F in the calulation of the CTFS, $\mathrm{X}[k]$, where

$$\mathrm{X}[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} \mathrm{x}(t) e^{-j2\pi(kf_F)t} dt \quad .$$

. What is the numerical value of $\mathrm{X}[-1]$?

$$\mathrm{X}_{CTFS}[k] = \frac{\mathrm{X}_{DFT}[k]}{N} \qquad \mathrm{X}_{CTFS}[-1] = \frac{X_{DFT}[-1]}{8}$$

$$\mathrm{X}_{CTFS}[-1] = \mathrm{X}^{*}_{CTFS}[1] = 4 - j2$$

$$\mathrm{X}_{CTFS}[-1] = \frac{1}{8}(4 - j2) = \frac{1}{2} - j\frac{1}{4}$$

(c) What is the average value of $\mathrm{x}(t)$?

The average value is $\mathrm{X}_{CTFS}[0]$ and $\mathrm{X}_{CTFS}[0] = \frac{\mathrm{X}_{DFT}[0]}{N} = \frac{4}{8} = \frac{1}{2}$.

(d) What is the total signal power of $\mathrm{x}(t)$?

The total signal power is the sum of the signal powers of all the complex exponentials that make up the signal. They are

$$\left|\frac{4}{8}\right|^2 + \left|\frac{4+j2}{8}\right|^2 + \left|\frac{-6+j4}{8}\right|^2 + \left|\frac{4-j10}{8}\right|^2 + \left|\frac{-2}{8}\right|^2 + + \left|\frac{4+j10}{8}\right|^2 + \left|\frac{-6-j4}{8}\right|^2 + \left|\frac{4-j2}{8}\right|^2$$

$$\frac{16 + (16+4) + (36+16) + (16+100) + 4 + (16+100) + (36+16) + (16+4)}{64} = 6.1875$$

21. For the two-dimensional spatial-domain function, $\mathrm{g}(x,y)$, below find its two-dimensional Fourier transform, $\mathrm{G}(f_x, f_y)$, and find the numerical value of the transform, $\mathrm{G}(f_x, f_y)$ at the spatial frequencies given. Then find the total signal energy in the signal under the definitions

$$E_g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathrm{g}(x,y)|^2 dxdy \text{ and } E_G = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathrm{G}(f_x, f_y)|^2 df_x df_y$$

for two-dimensional functions. (Parseval's theorem holds for two-dimensional functions also.)

$$\mathrm{g}(x,y) = 4\,\mathrm{tri}\left(\frac{x}{2}\right)\mathrm{rect}(10y)$$

$$\mathrm{G}(f_x, f_y) = 4 \times 2\,\mathrm{sinc}^2(2f_x) \times \frac{1}{10}\mathrm{sinc}\left(\frac{f_y}{10}\right) = \frac{4}{5}\mathrm{sinc}^2(2f_x)\,\mathrm{sinc}\left(\frac{f_y}{10}\right)$$

$$\mathrm{G}(0,1) = \frac{4}{5}\mathrm{sinc}^2(0)\,\mathrm{sinc}\left(\frac{1}{10}\right) = \frac{4}{5}\frac{\sin\left(\frac{\pi}{10}\right)}{\frac{\pi}{10}} = 0.7869$$

$$\mathrm{G}(0,10) = \frac{4}{5}\mathrm{sinc}^2(0)\,\mathrm{sinc}(1) = 0$$

$$E_g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|4\,\mathrm{tri}\left(\frac{x}{2}\right)\mathrm{rect}(10y)\right|^2 dx\,dy = 16\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{tri}^2\left(\frac{x}{2}\right)\mathrm{rect}^2(10y)\,dx\,dy$$

$$E_g = 16\left[\int_{-\infty}^{\infty}\mathrm{tri}^2\left(\frac{x}{2}\right)dx\int_{-\infty}^{\infty}\mathrm{rect}^2(10y)\,dy\right] = 16\left[\int_{-2}^{2}\mathrm{tri}^2\left(\frac{x}{2}\right)dx\int_{-\frac{1}{20}}^{\frac{1}{20}}dy\right]$$

$$E_g = \frac{16}{10}\left[2\int_0^2\mathrm{tri}^2\left(\frac{x}{2}\right)dx\right] = \frac{16}{5}\left[\int_0^2\left(1-\frac{x}{2}\right)^2 dx\right] = \frac{16}{5}\left[\int_0^2\left(1-x+\frac{x^2}{4}\right)dx\right]$$

$$E_g = \frac{16}{5}\left[x - \frac{x^2}{2} + \frac{x^3}{12}\right]_0^2 = \frac{32}{15} = 2.1333\ldots$$

22. A bandlimited, periodic, real signal, $\mathrm{x}(t)$, is sampled at its Nyquist rate eight times in one period. Let (one period of) the discrete Fourier transform of those samples be represented by the set,

$$\mathrm{DFT}[\{x\}] = \{X\} = \{a, b, c, d, 0, d^*, c^*, b^*\} \ .$$

That is, for this sampling, $\mathrm{X}[0] = a$, $\mathrm{X}[1] = b$,$\cdots$,$\mathrm{X}[7] = b^*$.

Answer the following questions in terms of the *a*, *b*, *c* and *d* of the original sampling.

a. What is the average value of the signal, $\mathrm{x}(t)$? $\frac{a}{8}$

b. What is the average signal power of $\mathrm{x}(t)$? (Assume there is no sinusoid in $\mathrm{x}(t)$ at exactly half the sampling rate.)

$$\left|\frac{a}{8}\right|^2 + 2\left|\frac{b}{8}\right|^2 + 2\left|\frac{c}{8}\right|^2 + 2\left|\frac{d}{8}\right|^2$$

c. If the signal, $\mathrm{x}(t)$, were sampled again over one period but at *twice the* Nyquist rate, what would (one period of) the DFT be?

$$\{2a,2b,2c,2d,0,0,0,0,0,0,0,0,0,2d^*,2c^*,2b^*\}$$

d. If the signal, $\mathrm{x}(t)$, were sampled again at the Nyquist rate but over *two* periods, what would (one period of) the DFT be?

$$\{2a,0,2b,0,2c,0,2d,0,0,0,2d^*,0,2c^*,0,2b^*,0\}$$

23. The discrete Fourier transform (DFT) of a set of three samples from a signal, $\mathrm{x}(t)$, is the set,

$$\{\mathrm{X}[0]\,,\,\mathrm{X}[1]\,,\,\mathrm{X}[2]\}=\left\{\,0\,,\,j\frac{3}{2}\,,-\,j\frac{3}{2}\right\}\ .$$

Find the three sample values.

$$\mathrm{x}[n]=\frac{1}{N_F}\sum_{k=0}^{N_F-1}\mathrm{X}[k]e^{j2\pi\frac{nk}{N_F}}$$

$$\mathrm{x}[0]=\frac{1}{3}\sum_{k=0}^{2}\mathrm{X}[k]=\frac{1}{3}\left[0+j\frac{3}{2}-j\frac{3}{2}\right]=0$$

$$\mathrm{x}[1]=\frac{1}{3}\sum_{k=0}^{2}\mathrm{X}[k]e^{j\frac{2\pi n}{3}}=\frac{1}{3}\left[0+j\frac{3}{2}e^{j\frac{2\pi}{3}}-j\frac{3}{2}e^{j\frac{4\pi}{3}}\right]$$

$$=\frac{1}{3}\left\{0+j\frac{3}{2}\left[\cos\left(\frac{2\pi}{3}\right)+j\sin\left(\frac{2\pi}{3}\right)\right]-j\frac{3}{2}\left[\cos\left(\frac{4\pi}{3}\right)+j\sin\left(\frac{4\pi}{3}\right)\right]\right\}=-\frac{\sqrt{3}}{2}$$

$$\mathrm{x}[2]=\frac{1}{3}\sum_{k=0}^{2}\mathrm{X}[k]e^{j\frac{4\pi n}{3}}=\frac{1}{3}\left[0+j\frac{3}{2}e^{j\frac{4\pi}{3}}-j\frac{3}{2}e^{j\frac{8\pi}{3}}\right]$$

$$=\frac{1}{3}\left\{0+j\frac{3}{2}\left[\cos\left(\frac{4\pi}{3}\right)+j\sin\left(\frac{4\pi}{3}\right)\right]-j\frac{3}{2}\left[\cos\left(\frac{8\pi}{3}\right)+j\sin\left(\frac{8\pi}{3}\right)\right]\right\}=\frac{\sqrt{3}}{2}$$

24. Given two signals,

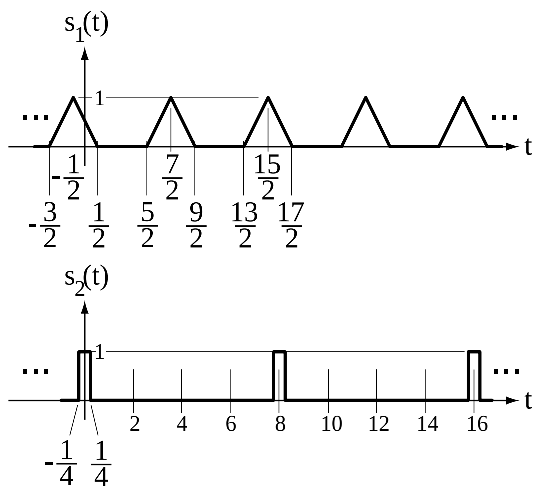
$$\mathrm{x}_1(t)=\mathrm{tri}\left(t+\frac{1}{2}\right)*\frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right)\ \text{ and }\ \mathrm{x}_2(t)=\mathrm{rect}(2t)*\frac{1}{8}\mathrm{comb}\left(\frac{t}{8}\right)\ ,$$

find at least one value of "τ" at which the crosscorrelation function, $R_{12}(\tau)$, is

a. a maximum (most positive value),
b. a minimum (most negative value),
c. zero. (If the crosscorrelation is never zero just state that fact.)

$$\mathrm{x}_1(t)=\mathrm{tri}\left(t+\frac{1}{2}\right)*\sum_{n=-\infty}^{\infty}\delta(t-4n)\ \text{ and }\ \mathrm{x}_2(t)=\mathrm{rect}(2t)*\sum_{n=-\infty}^{\infty}\delta(t-8n)$$

$$x_1(t) = \sum_{n=-\infty}^{\infty} \text{tri}\left(t - 4n + \frac{1}{2}\right) \text{ and } x_2(t) = \sum_{n=-\infty}^{\infty} \text{rect}[2(t - 8n)]$$



a. The maximum crosscorrelation will occur when the peaks line up and that will be at a shift of

$$\tau = -\frac{1}{2} \pm 4n \quad , \quad n \text{ an integer}$$

b. Since the two signals are both always non-negative, the minimum crosscorrelation will be zero and will occur whenever the non-zero portions of the two signals do not overlap. This condition would occur at shifts of

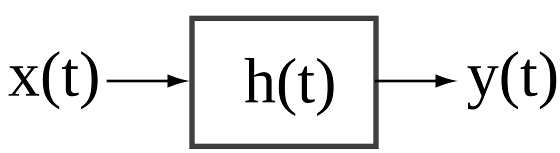
$$\frac{3}{4} \pm 2n < \tau < \frac{9}{4} \pm 2n \quad , \quad n \text{ an integer}$$

c. The crosscorrelation iis zero at the shifts given in part b.

25. For the system below, the transfer function, H(s), is given by

$$\text{H}(s) = \frac{\text{Y}(s)}{\text{X}(s)} = L[\text{h}(t)] = \frac{s^2}{(s+1)(s+2)(s+3)} \quad .$$

Assuming that the system is initially at rest (nothing has happened for all negative time), find the step response of this system. What are the value and the first derivative of the step response at time, $t = 0^+$? Verify these values using the initial value theorem of the Laplace transform.



$$\text{X}(s) = \frac{1}{s} \quad , \quad \text{Y}(s) = \frac{1}{s}\frac{s^2}{(s+1)(s+2)(s+3)} = \frac{s}{(s+1)(s+2)(s+3)}$$

$$\mathrm{Y}(s) = -\frac{\frac{1}{2}}{s+1} + \frac{2}{s+2} - \frac{\frac{3}{2}}{s+3}$$

$$\mathrm{y}(t) = -\frac{1}{2}e^{-t} + 2e^{-2t} - \frac{3}{2}e^{-3t}$$

$$\mathrm{y}(0^+) = -\frac{1}{2} + 2 - \frac{3}{2} = 0$$

$$\lim_{s\to\infty} s\,\mathrm{Y}(s) = \frac{s^2}{(s+1)(s+2)(s+3)} = 0 \quad \text{Check}$$

$$\frac{d}{dt}[\mathrm{y}(t)] = \frac{1}{2}e^{-t} - 4e^{-2t} + \frac{9}{2}e^{-3t}$$

$$\frac{d}{dt}[\mathrm{y}(t)]_{t\to 0^+} = \frac{1}{2} - 4 + \frac{9}{2} = 1$$

$$\lim_{s\to\infty} s^2\,\mathrm{Y}(s) = \frac{s^3}{(s+1)(s+2)(s+3)} = 1 \quad \text{Check}$$

26. An analog filter has the Laplace-domain transfer function,

$$\mathrm{H}_s(s) = \frac{1}{s+10} \quad .$$

Find the recursion relation for a digital filter whose step response consists of samples of the step response of the analog filter. Let the time between samples, T_s, be 0.1 seconds.

The step response is

$$\mathrm{h}_{-1}(t) = L^{-1}\left[\frac{\mathrm{H}_s(s)}{s}\right] \quad .$$

The z transform of this step response is

$$Z[\mathrm{h}_{-1}(t)]$$

which must equal the z domain step response form

$$\frac{z}{z-1}\mathrm{H}_z(z)$$

where $\mathrm{H}_z(z)$ is the z domain transfer function. That is,

$$\mathrm{H}_z(z) = \frac{z-1}{z} Z\left\{L^{-1}\left[\frac{\mathrm{H}_s(s)}{s}\right]\right\} \; .$$

$$\mathrm{H}_z(z) = \frac{z-1}{z} Z\left\{L^{-1}\left[\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}\right]\right\} = \frac{z-1}{z} Z\left\{\frac{1}{10}\mathrm{u}(t) - \frac{1}{10}e^{-10t}\mathrm{u}(t)\right\}$$

$$\mathrm{H}_z(z) = \frac{z-1}{z}\left[\frac{1}{10}\frac{z}{z-1} - \frac{1}{10}\frac{z}{z-e^{-10T_s}}\right] = \frac{1}{10}\left[1 - \frac{z-1}{z-e^{-10T_s}}\right]$$

$$\mathrm{H}_z(z) = \frac{1}{10}\frac{1-e^{-1}}{z-e^{-1}} = \frac{0.0632}{z-0.368}$$

$$(z-0.368)\,\mathrm{Y}_z(z) = 0.0632\,\mathrm{X}_z(z)$$

$$\mathrm{y}[n+1] - 0.368\,\mathrm{y}[n] = 0.0632\,\mathrm{x}[n]$$

$$\mathrm{y}[n] = 0.0632\,\mathrm{x}[n-1] + 0.368\,\mathrm{y}[n-1]$$

27. Find all values of z which satisfy the equation,

$$je^z = -1$$

$$je^z = je^x[\cos(y) + j\sin(y)] = je^x\cos(y) - e^x\sin(y) = -1$$

$$e^x\cos(y) = 0 \quad \text{and} \quad -e^x\sin(y) = -1$$

$$e^x\cos(y) = 0 \Rightarrow x \to -\infty \quad \text{or} \quad y = \frac{\pi}{2} + n\pi$$

$$e^x\sin(y) = 1 \Rightarrow e^x(\pm 1) = 1 \Rightarrow x > -\infty \quad \text{and} \quad y = \frac{\pi}{2} + 2n\pi$$

$$e^x = 1 \Rightarrow x = 0$$

$$z = j\left(\frac{\pi}{2} + 2n\pi\right)$$

Check:

$$je^{j\left(\frac{\pi}{2}+2n\pi\right)} = -1$$

$$\cos\left(\frac{\pi}{2} + 2n\pi\right) + j\sin\left(\frac{\pi}{2} + 2n\pi\right) = j$$

$$1 = 1$$

28. Find the residue of

$$g(z) = \frac{\sqrt{z}}{z^3 - 1}$$

at the pole which lies in the third quadrant (where $x < 0$ and $y < 0$) and on the branch of the square root function,

$$0 < \theta < 2\pi \quad .$$

$$g(z) = \frac{\sqrt{z}}{(z-1)\left(z + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(z + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}$$

So the poles are at

$$z = 1 \quad , \quad z = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \quad , \quad \underbrace{z = -\frac{1}{2} - j\frac{\sqrt{3}}{2}}_{\text{third quadrant}}$$

$$\text{residue} = \frac{\sqrt{-\frac{1}{2} - j\frac{\sqrt{3}}{2}}}{\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2} - 1\right)\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2} + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)} = \frac{\sqrt{e^{j\frac{4\pi}{3}}}}{\left(-\frac{3}{2} - j\frac{\sqrt{3}}{2}\right)\left(-j\sqrt{3}\right)}$$

$$\text{residue} = \frac{2e^{j\frac{2\pi}{3}}}{\left(-3 - j\sqrt{3}\right)\left(-j\sqrt{3}\right)} = \frac{1}{3} \times \frac{-1 + j\sqrt{3}}{-1 + j\sqrt{3}} = \frac{1}{3}$$

29. Find the two-dimensional Fourier transform, $G(f_x, f_y)$, of the two-dimensional space-domain function,

$$g(x,y) = 4\,\text{tri}(2x)\,\text{sinc}\left(\frac{y-1}{3}\right) \quad .$$

Then sketch the one-dimensional "cross-section",

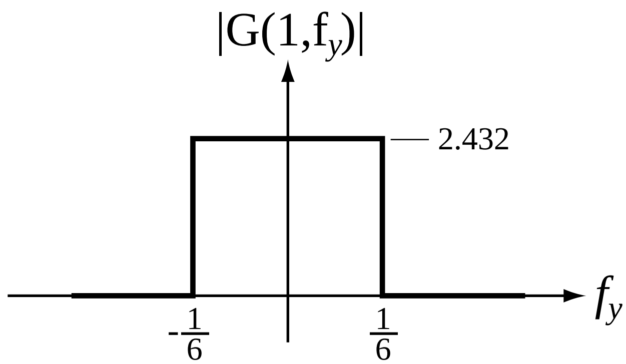
$$\left|G(1, f_y)\right| \quad ,$$

of the magnitude of the spatial-frequency domain function.

$$G(f_x, f_y) = 6\,\text{sinc}^2\left(\frac{f_x}{2}\right)\text{rect}(3f_y)\,e^{-j2\pi f_y}$$

$$G(1, f_y) = 6\,\text{sinc}^2\left(\frac{1}{2}\right)\text{rect}(3f_y)e^{-j2\pi f_y} = 2.432\,\text{rect}(3f_y)e^{-j2\pi f_y}$$

$$\left|G(1, f_y)\right| = 2.432\,\text{rect}(3f_y)$$



30. Find the numerical value of the contour integral,

$$I = \int_C 2zz^* dz$$

if the contour C is a straight line from the point, $z = 1$ to the point $z = 3 + j2$.

On the contour, $y = x - 1$. Therefore

$$z = x + jy = x + j(x-1) = x(1+j) - j$$

$$dz = (1+j)dx$$

$$I = \int_1^3 2[x + j(x-1)][x - j(x-1)](1+j)dx = 2(1+j)\int_1^3 [x(1+j) - j][x(1-j) + j]dx$$

$$I = 2(1+j)\int_1^3 [2x^2 - 2x + 1]dx = 2(1+j)\left[\frac{2}{3}x^3 - x^2 + x\right]_1^3$$

$$I = \frac{68}{3}(1+j) = 32\angle 45°$$

31. A linear system has an impulse response, $h(t) = e^{-\alpha t}u(t)$. Find its response, $y(t)$, to an excitation which is described by

$$x(t) = \begin{cases} 1 \ , & 0 < t < 1 \\ 0 \ , & \text{elsewhere} \end{cases} .$$

$$\mathrm{y}(t)=\mathrm{x}(t)*\mathrm{h}(t)=\int_{-\infty}^{\infty}\mathrm{x}(\tau)\mathrm{h}(t-\tau)d\tau=\int_{0}^{1}e^{-\alpha(t-\tau)}\mathrm{u}(t-\tau)d\tau$$

For $t<0$, $\mathrm{y}(t)=0$.

For $0<t<1$,

$$\mathrm{y}(t)=\int_{0}^{t}e^{-\alpha(t-\tau)}d\tau=\left.\frac{e^{-\alpha(t-\tau)}}{\alpha}\right|_{0}^{t}=\frac{1-e^{-\alpha t}}{\alpha}$$

For $t>1$,

$$\mathrm{y}(t)=\int_{0}^{1}e^{-\alpha(t-\tau)}d\tau=\left.\frac{e^{-\alpha(t-\tau)}}{\alpha}\right|_{0}^{1}=\frac{e^{-\alpha(t-1)}-e^{-\alpha t}}{\alpha}$$

32. Find the inverse Fourier transform, $\mathrm{x}(t)$, of the frequency-domain function,

$$\mathrm{X}(f)=4\left[\mathrm{rect}\left(\frac{f-100}{500}\right)+\mathrm{rect}\left(\frac{f+100}{500}\right)\right]e^{-j\frac{\pi f}{500}}$$

Express the result, $\mathrm{x}(t)$, as a function of time with real coefficients and real-valued functions.

$$\mathrm{sinc}(t)\xleftrightarrow{F}\mathrm{rect}(f)$$

$$500\,\mathrm{sinc}(500t)\xleftrightarrow{F}\mathrm{rect}\left(\frac{f}{500}\right)$$

$$500\,\mathrm{sinc}(500t)e^{+j200\pi t}\xleftrightarrow{F}\mathrm{rect}\left(\frac{f-100}{500}\right)$$

$$500\,\mathrm{sinc}(500t)e^{+j200\pi t}+500\,\mathrm{sinc}(500t)e^{-j200\pi t}\xleftrightarrow{F}\mathrm{rect}\left(\frac{f-100}{500}\right)+\mathrm{rect}\left(\frac{f+100}{500}\right)$$

$$500\,\mathrm{sinc}\left[500\left(t-\frac{1}{1000}\right)\right]\left[e^{+j200\pi\left(t-\frac{1}{1000}\right)}+e^{-j200\pi\left(t-\frac{1}{1000}\right)}\right]\xleftrightarrow{F}\left[\mathrm{rect}\left(\frac{f-100}{500}\right)+\mathrm{rect}\left(\frac{f+100}{500}\right)\right]e^{-j\frac{\pi f}{500}}$$

$$2000\,\mathrm{sinc}\left[500\left(t-\frac{1}{1000}\right)\right]\left[e^{+j200\pi\left(t-\frac{1}{1000}\right)}+e^{-j200\pi\left(t-\frac{1}{1000}\right)}\right]\xleftrightarrow{F}4\left[\mathrm{rect}\left(\frac{f-100}{500}\right)+\mathrm{rect}\left(\frac{f+100}{500}\right)\right]e^{-j\frac{\pi f}{500}}$$

$$4000\,\mathrm{sinc}\left[500\left(t-\frac{1}{1000}\right)\right]\cos\left[200\pi\left(t-\frac{1}{1000}\right)\right]\xleftrightarrow{F}4\left[\mathrm{rect}\left(\frac{f-100}{500}\right)+\mathrm{rect}\left(\frac{f+100}{500}\right)\right]e^{-j\frac{\pi f}{500}}$$

33. Find the numerical value of the contour integral,

$$I = \int_C \left(2z - 3z^*\right) dz$$

if the contour C is a straight line from the point, $z = 0$ to the point $z = 4 + j5$.

$$z = x + jy \text{ and } z^* = x - jy$$

On the contour, $5x = 4y$. Therefore on the contour, $z = \left(1 + j\frac{5}{4}\right)x$ and $z^* = \left(1 - j\frac{5}{4}\right)x$ and $dz = \left(1 + j\frac{5}{4}\right)dx$. Therefore

$$I = \int_C \left(2z - 3z^*\right) dz = \int_0^4 \left[2\left(1 + j\frac{5}{4}\right)x - 3\left(1 - j\frac{5}{4}\right)x\right]\left(1 + j\frac{5}{4}\right)dx$$

$$I = \int_0^4 \left(-1 + j\frac{25}{4}\right)x\left(1 + j\frac{5}{4}\right)dx = \left(\frac{141}{16} + j5\right)\int_0^4 x\,dx$$

$$I = \left(\frac{141}{16} + j5\right)\left[\frac{x^2}{2}\right]_0^4 = \left(\frac{141}{16} + j5\right)8 = \frac{141}{2} + j40$$

34. Find the numerical values of the following residues:

(a) Residue of $\frac{z}{z^2 - 1}$ at $z = -1$.

$$\text{Residue} = \frac{-1}{(-1-1)} = \frac{1}{2}$$

(b) Residue of $\frac{\log(z)}{z^4 + 1}$ at $z = \frac{-1-j}{\sqrt{2}}$ on the branch, $-\pi < \theta < \pi$ where θ is the angle of z

$$\frac{\log(z)}{z^4 + 1} = \frac{\log(z)}{\left(z - \frac{1+j}{\sqrt{2}}\right)\left(z - \frac{1-j}{\sqrt{2}}\right)\left(z - \frac{-1+j}{\sqrt{2}}\right)\left(z - \frac{-1-j}{\sqrt{2}}\right)}$$

$$\text{Residue} = \frac{\log\left(\frac{-1-j}{\sqrt{2}}\right)}{\left(\frac{-1-j}{\sqrt{2}} - \frac{1+j}{\sqrt{2}}\right)\left(\frac{-1-j}{\sqrt{2}} - \frac{1-j}{\sqrt{2}}\right)\left(\frac{-1-j}{\sqrt{2}} - \frac{-1+j}{\sqrt{2}}\right)}$$

$$\text{Residue} = \frac{-j\dfrac{3\pi}{4}}{\left(2\dfrac{-1-j}{\sqrt{2}}\right)\left(\dfrac{-2}{\sqrt{2}}\right)\left(\dfrac{-j2}{\sqrt{2}}\right)} = \frac{3\pi}{16}\frac{\sqrt{2}}{1+j} = \frac{3\pi}{16\sqrt{2}}(1-j) = 0.589\angle-45^\circ$$

(c) Residue of $\dfrac{\sqrt[3]{z}}{z^3-8}$ at $z=-1-j\sqrt{3}$ on the branch, $0<\theta<2\pi$ where θ is the angle of z

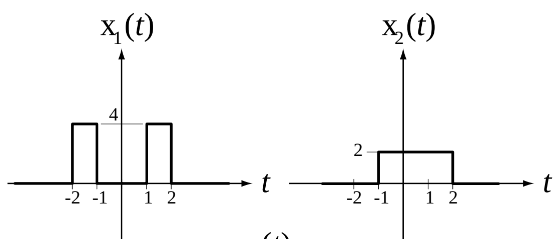
$$\frac{\sqrt[3]{z}}{z^3-8} = \frac{\sqrt[3]{z}}{(z-2)\left(z+1-j\sqrt{3}\right)\left(z+1+j\sqrt{3}\right)}$$

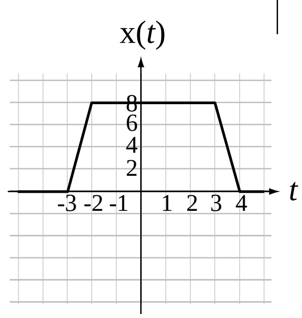
$$\text{Residue} = \frac{\sqrt[3]{-1-j\sqrt{3}}}{\left(-1-j\sqrt{3}-2\right)\left(-1-j\sqrt{3}+1-j\sqrt{3}\right)}$$

$$\text{Residue} = \frac{\sqrt[3]{2\angle\dfrac{4\pi}{3}}}{j2\sqrt{3}\left(3+j\sqrt{3}\right)} = \frac{\sqrt[3]{2}\angle\dfrac{4\pi}{9}}{j2\sqrt{3}\left(3+j\sqrt{3}\right)} = \frac{1.26\angle\dfrac{4\pi}{9}}{j2\sqrt{3}\left(3+j\sqrt{3}\right)} = \frac{0.2189+j1.241}{6\left(1-j\sqrt{3}\right)}$$

$$\text{Residue} = \frac{1.26\angle80^\circ}{6\times2\angle-60^\circ} = 0.105\angle140^\circ = -0.0804+j0.0675$$

35. In the space provided sketch the convolution, $x(t)$, of the two signals, $x_1(t)$ and $x_2(t)$ illustrated below. The sketch should show the general shape and should have a scale so that the values of t and $x(t)$ could be read from the sketch (approximately).





36. A linear system has an excitation, $x(t)$, and a response, $y(t)$. It is characterized by its impulse response, $h(t) = 10\,\text{sinc}\left[10^7\left(t-10^{-6}\right)\right]$. Find a real, time-domain expression for the system response, $y(t)$, if the system excitation is

(a) $\mathrm{x}(t) = 100\cos(40\pi \times 10^6 t)$

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = F^{-1}\{F[\mathrm{x}(t)]F[\mathrm{h}(t)]\}$$

$$F[\mathrm{x}(t)] = 50[\delta(f - 20\mathrm{MHz}) + \delta(f + 20\mathrm{MHz})]$$

$$F[\mathrm{h}(t)] = 10^{-6}\,\mathrm{rect}\left(\frac{f}{10\mathrm{MHz}}\right)e^{-j2\pi\times 10^{-6} f}$$

$$F[\mathrm{x}(t)]F[\mathrm{h}(t)] = 0$$

$$\therefore\ \mathrm{y}(t) = 0$$

(b) $\mathrm{x}(t) = (4\times 10^6)\,\mathrm{comb}(4\times 10^6 t)$

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = F^{-1}\{F[\mathrm{x}(t)]F[\mathrm{h}(t)]\}$$

$$F[\mathrm{x}(t)] = \mathrm{comb}\left(\frac{f}{4\mathrm{MHz}}\right)$$

$$F[\mathrm{h}(t)] = 10^{-6}\,\mathrm{rect}\left(\frac{f}{10\mathrm{MHz}}\right)e^{-j2\pi\times 10^{-6} f}$$

$$F[\mathrm{h}(t)]F[\mathrm{x}(t)] = \mathrm{comb}\left(\frac{f}{4\mathrm{MHz}}\right)\times 10^{-6}\,\mathrm{rect}\left(\frac{f}{10\mathrm{MHz}}\right)e^{-j2\pi\times 10^{-6} f}$$

$$F[\mathrm{h}(t)]F[\mathrm{x}(t)] = 4\sum_{n=-\infty}^{\infty}\delta(f - 4\times 10^6 n)\,\mathrm{rect}\left(\frac{f}{10\mathrm{MHz}}\right)e^{-j2\pi\times 10^{-6} f}$$

$$F[\mathrm{h}(t)]F[\mathrm{x}(t)] = 4\sum_{n=-1}^{1}\delta(f - 4\times 10^6 n)e^{-j2\pi\times 10^{-6} f}$$

$$F[\mathrm{h}(t)]F[\mathrm{x}(t)] = 4\left[\delta(f) + \delta(f - 4\mathrm{MHz})\underbrace{e^{-j8\pi}}_{=1} + \delta(f + 4\mathrm{MHz})\underbrace{e^{+j8\pi}}_{=1}\right]$$

$$\mathrm{y}(t) = F^{-1}\{F[\mathrm{h}(t)]F[\mathrm{x}(t)]\} = 4 + 8\cos(8\pi\times 10^6 t)$$

37. An entire function is one which is analytic at every point in the complex plane. Is the function, $f(z) = e^z\cos(z)$, an entire function? Using the Cauchy-Riemann conditions show that it is, or is not, entire.

The Cauchy-Riemann conditions are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$f(z)=e^z\cos(z)=e^z\frac{e^{jz}+e^{-jz}}{2}=e^{x+jy}\frac{e^{j(x+jy)}+e^{-j(x+jy)}}{2}$$

$$f(z)=\frac{e^{x+jx-y+jy}+e^{x-jx+y+jy}}{2}=\frac{e^{x-y}e^{j(x+y)}+e^{x+y}e^{-j(x-y)}}{2}$$

$$f(z)=\frac{1}{2}\left\{e^{x-y}\left[\cos(x+y)+j\sin(x+y)\right]+e^{x+y}\left[\cos(x-y)-j\sin(x-y)\right]\right\}$$

$$u=\frac{1}{2}\left\{e^{x-y}\left[\cos(x+y)\right]+e^{x+y}\left[\cos(x-y)\right]\right\} \text{ and } v=\frac{1}{2}\left\{e^{x-y}\left[\sin(x+y)\right]-e^{x+y}\left[\sin(x-y)\right]\right\}$$

$$\frac{\partial u}{\partial x}=\frac{1}{2}\left\{-e^{x-y}\sin(x+y)+e^{x-y}\cos(x+y)-e^{x+y}\sin(x-y)+e^{x+y}\cos(x-y)\right\}$$
$$\frac{\partial v}{\partial y}=\frac{1}{2}\left\{e^{x-y}\cos(x+y)-e^{x-y}\sin(x+y)+e^{x+y}\cos(x-y)-e^{x+y}\sin(x-y)\right\}$$

$$\frac{\partial u}{\partial y}=\frac{1}{2}\left\{-e^{x-y}\sin(x+y)-e^{x-y}\cos(x+y)+e^{x+y}\sin(x-y)+e^{x+y}\cos(x-y)\right\}$$
$$\frac{\partial v}{\partial x}=\frac{1}{2}\left\{e^{x-y}\cos(x+y)+e^{x-y}\sin(x+y)-e^{x+y}\cos(x-y)-e^{x+y}\sin(x-y)\right\}$$

The Cauchy-Riemann conditions are satisfied everywhere in the complex plane so this is an entire function.

38. Evaluate the real integral,

$$I=\int_{-\infty}^{\infty}\frac{xdx}{(x^2+4x+8)(x^2-2x+5)}$$

using contour integration in the complex plane and the method of residues.

$$I=\int_{-\infty}^{\infty}\frac{xdx}{(x^2+4x+8)(x^2-2x+5)}=\int_{-\infty}^{\infty}\frac{xdx}{(x+2-j2)(x+2+j2)(x-1-j2)(x-1+j2)}$$

$$I=\oint_{C,ccw}\frac{zdz}{(z+2-j2)(z+2+j2)(z-1-j2)(z-1+j2)}$$

where the contour is the real axis plus an infinite-radius semicircle in the upper half-plane.

$$I=j2\pi\sum \text{Residues at } z=-2+j2 \text{ and } z=1+j2$$

$$I = j2\pi\left[\frac{-2+j2}{(j4)(-3)(-3+j4)} + \frac{1+j2}{3(3+j4)(j4)}\right] = \pi\left[\frac{-2+j2}{6(3-j4)} + \frac{1+j2}{6(3+j4)}\right]$$

$$I = \pi\left[\frac{(-2+j2)(3+j4)+(1+j2)(3-j4)}{6(3-j4)(3+j4)}\right] = -\frac{\pi}{50}$$

39. A simplified version of a type of signal which occurs in binary phase shift keying is given by

$$\mathrm{x}(t) = \sin(2\pi t)\left\{\left[2\,\mathrm{rect}(t) * \mathrm{comb}\left(\frac{t}{2}\right)\right] - 1\right\} \ .$$

Sketch the magnitude of the Fourier transform, S(f), of s(t). The sketch should have a scale on both the frequency and magnitude axes so that rough approximations to the functional values could be read from the sketch.

$$\mathrm{X}(f) = \frac{j}{2}\left[\delta(f+1) - \delta(f-1)\right] * \left\{\left[2\,\mathrm{sinc}(f) 2\,\mathrm{comb}(2f)\right] - \delta(f)\right\}$$

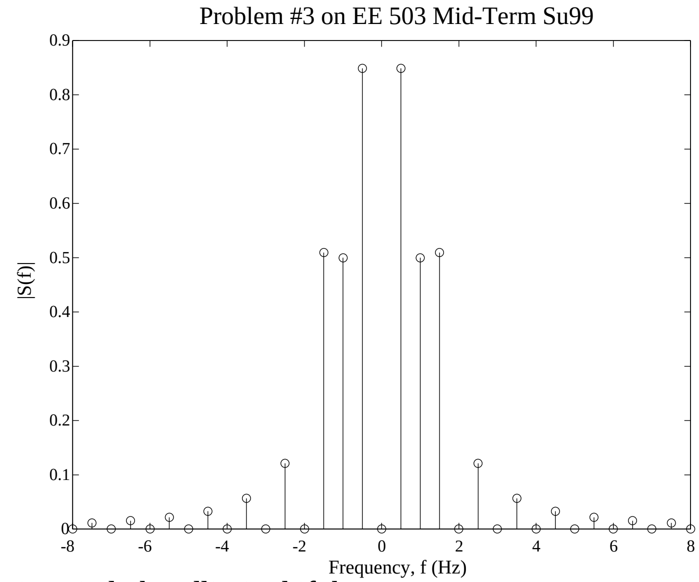
$$\mathrm{X}(f) = \frac{j}{2}\left\{4\left[\delta(f+1) - \delta(f-1)\right] * \mathrm{sinc}(f)\mathrm{comb}(2f) - \delta(f+1) + \delta(f-1)\right\}$$

$$\mathrm{X}(f) = \frac{j}{2}\left\{4\left[\mathrm{sinc}(f+1)\mathrm{comb}\left(2(f+1)\right) - \mathrm{sinc}(f-1)\mathrm{comb}\left(2(f-1)\right)\right] - \delta(f+1) + \delta(f-1)\right\}$$

$$\mathrm{X}(f) = \frac{j}{2}\left\{4\left[\mathrm{sinc}(f+1)\sum_{n=-\infty}^{\infty}\delta\left[2(f+1) - n\right] - \mathrm{sinc}(f-1)\sum_{n=-\infty}^{\infty}\delta\left[2(f-1) - n\right]\right] - \delta(f+1) + \delta(f-1)\right\}$$

$$\mathrm{X}(f) = \frac{j}{2}\left\{2\left[\mathrm{sinc}(f+1)\sum_{n=-\infty}^{\infty}\delta\left(f+1-\frac{n}{2}\right) - \mathrm{sinc}(f-1)\sum_{n=-\infty}^{\infty}\delta\left(f-1-\frac{n}{2}\right)\right] - \delta(f+1) + \delta(f-1)\right\}$$

$$\mathrm{X}(f) = \frac{j}{2}\left\{2\left[\sum_{n=-\infty}^{\infty}\mathrm{sinc}\left(\frac{n}{2}\right)\delta\left(f+1-\frac{n}{2}\right) - \sum_{n=-\infty}^{\infty}\mathrm{sinc}\left(\frac{n}{2}\right)\delta\left(f-1-\frac{n}{2}\right)\right] - \delta(f+1) + \delta(f-1)\right\}$$



40. A function is strictly bandlimited if there are no non-zero components in its Fourier transform for frequencies whose magnitudes are above some finite value. This applies also to two-dimensional functions in both spatial-frequency dimensions. That is, a space-domain function is strictly bandlimited if there are no non-zero components in its two-dimensional Fourier transform for spatial frequencies whose magnitudes are above some finite value. A two-dimensional function may be bandlimited in both dimensions, not bandlimited in both dimensions or bandlimited in one dimension and not bandlimited in the other dimension.

Find the two-dimensional Fourier transform, $\mathrm{G}\left(f_x, f_y\right)$, of

$$\mathrm{g}(x,y) = \left[\mathrm{sinc}\left(\frac{x+1}{3}\right) * \frac{1}{8}\mathrm{comb}\left(\frac{x}{8}\right)\right]\cos(2\pi y) \ ,$$

determine whether or not it is strictly bandlimited in each dimension and, if it is strictly bandlimited in either or both dimensions, find the bandwidth limit in each dimension in which it is bandlimited.

$$\mathrm{G}\left(f_x, f_y\right) = \left[3\mathrm{rect}(3f_x)e^{-j2\pi f}\ \mathrm{comb}(8f_x)\right]\frac{1}{2}\left[\delta\left(f_y - 1\right) + \delta\left(f_y + 1\right)\right]$$

$$\mathrm{G}\left(f_x, f_y\right) = \frac{3}{16}\left[e^{-j2\pi f}\sum_{n=-1}^{1}\delta\left(f_x - \frac{n}{8}\right)\right]\left[\delta\left(f_y - 1\right) + \delta\left(f_y + 1\right)\right]$$

It is bandlimited in both dimensions. In the x dimension the bandlimit is 1/8 and in the y dimension the bandlimit is 1.

41. (a) Let the branch cut of the function,

$$w = \frac{z^{\frac{1}{2}}}{z^2 + 1} \ ,$$

be the negative real axis with the principal branch being defined by the angular range, $-\pi < \theta \le \pi$ where $z = re^{j\theta}$. What is the residue of this function at $z = -j$ on the principal branch?

For the principal branch, $-\pi < \theta \le \pi$:

The residue at $z = -j$ is found by using the procedure,

$$\text{Residue at } z = -j \text{ is } \left[(z+j)\frac{z^{\frac{1}{2}}}{(z+j)(z-j)} \right]_{z=-j}$$

or

$$\text{Residue at } z = -j \text{ is } \left[\frac{(-j)^{\frac{1}{2}}}{(-j-j)} \right]_{z=-j} .$$

On the principal branch, $-\pi < \theta \le \pi$, $-j = e^{-j\frac{\pi}{2}}$ and $(-j)^{\frac{1}{2}} = \left(e^{-j\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{-j\frac{\pi}{4}}$

and

$$\text{Residue at } z = -j \text{ is } \frac{e^{-j\frac{\pi}{4}}}{(-j-j)} = \frac{\frac{1-j}{\sqrt{2}}}{-j2} = \frac{1+j}{2\sqrt{2}} \text{ or } \frac{e^{+j\frac{\pi}{4}}}{2} .$$

(b) If the branch cut is changed to the positive real axis and the principal branch is defined by the angular range, $0 < \theta \le 2\pi$, what is the residue at $z = -j$ on this principal branch?

For the principal branch, $0 < \theta \le 2\pi$:

The residue at $z = -j$ is found, as before, by using the procedure,

$$\text{Residue at } z = -j \text{ is } \left[\frac{(-j)^{\frac{1}{2}}}{(-j-j)} \right]_{z=-j} .$$

For the principal branch, $0 < \theta \le 2\pi, -j = e^{+j\frac{3\pi}{2}}$ and $(-j)^{\frac{1}{2}} = \left(e^{+j\frac{3\pi}{2}}\right)^{\frac{1}{2}} = e^{+j\frac{3\pi}{4}}$

and

$$\text{Residue at } z = -j = \frac{e^{+j\frac{3\pi}{4}}}{(-j-j)} = \frac{\frac{-1+j}{\sqrt{2}}}{-j2} = -\frac{1+j}{2\sqrt{2}} \text{ or } \frac{e^{+j\frac{5\pi}{4}}}{2} .$$

42. For the function,

$$w = \frac{1}{1-z},$$

evaluate the Cauchy-Riemann conditions and describe the regions in the "z" plane in which it is analytic and the regions in the "z" plane in which it is not analytic.

The Cauchy-Riemann conditions are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $z = z + jy$ and $w = u + jv$.

$$u + jv = \frac{1}{1-x-jy} = \frac{1}{1-x-jy}\frac{1-x+jy}{1-x+jy} = \frac{1-x+jy}{(1-x)^2+y^2}$$

$$u = \frac{1-x}{(1-x)^2+y^2} \text{ and } v = \frac{y}{(1-x)^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\left[(1-x)^2+y^2\right](-1)-(1-x)\left[-2(1-x)\right]}{\left[(1-x)^2+y^2\right]^2} = \frac{(1-x)^2-y^2}{\left[(1-x)^2+y^2\right]^2}$$

$$\frac{\partial v}{\partial y} = \frac{(1-x)^2+y^2-y(2y)}{\left[(1-x)^2+y^2\right]^2} = \frac{(1-x)^2-y^2}{\left[(1-x)^2+y^2\right]^2}$$

$$\frac{\partial u}{\partial y} = \frac{-(1-x)2y}{\left[(1-x)^2+y^2\right]^2} = -\frac{2y(1-x)}{\left[(1-x)^2+y^2\right]^2}$$

$$\frac{\partial v}{\partial x} = \frac{2y(1-x)}{\left[(1-x)^2+y^2\right]^2}$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every point in the complex plane except the point $z = 1$, the function is analytic everywhere in the complex plane except at the point, $z = 1$.

43. If $\mathrm{G}(f) = \left[\mathrm{rect}\left(\frac{f-20}{4}\right) + \mathrm{rect}\left(\frac{f+20}{4}\right)\right]e^{-j\frac{\pi f}{2}}$ find the inverse Fourier transform, $\mathrm{g}(t) = F^{-1}\left[\mathrm{G}(f)\right]$ and express it as simply as you can in terms of entirely real functions.

(Using $\frac{1}{|a|}\mathrm{g}\left(\frac{t}{a}\right) \xleftrightarrow{F} \mathrm{G}(af)$, $\mathrm{rect}\left(\frac{f}{4}\right) \xleftrightarrow{F} 4\,\mathrm{sinc}(4t)$.

Using $e^{j2\pi f_0 t}\mathrm{g}(t) \xleftrightarrow{F} \mathrm{G}(f - f_0)$, $\mathrm{rect}\left(\frac{f-20}{4}\right) \xleftrightarrow{F} 4\,\mathrm{sinc}(4t)e^{j40\pi t}$.

Using $\alpha g(t) + \beta h(t) \leftrightarrow \alpha G(f) + \beta H(f)$,

$$\mathrm{rect}\left(\frac{f-20}{4}\right) + \mathrm{rect}\left(\frac{f+20}{4}\right) \xleftrightarrow{F} 4\,\mathrm{sinc}(4t)e^{j40\pi t} + 4\,\mathrm{sinc}(4t)e^{-j40\pi t}$$

Using $\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$,

$$\mathrm{rect}\left(\frac{f-20}{4}\right) + \mathrm{rect}\left(\frac{f+20}{4}\right) \xleftrightarrow{F} 8\,\mathrm{sinc}(4t)\cos(40\pi t)$$

Using $\mathrm{g}(t - t_0) \leftrightarrow \mathrm{G}(f)e^{-j2\pi f t_0}$,

$$\left[\mathrm{rect}\left(\frac{f-20}{4}\right) + \mathrm{rect}\left(\frac{f+20}{4}\right)\right]e^{-j\frac{\pi f}{2}}\ \mathrm{g}(t) 8\,\mathrm{sinc}\left[4\left(t - \frac{1}{4}\right)\right]\cos\left[40\pi\left(t - \frac{1}{4}\right)\right]$$

Using $\cos(x \pm 2n\pi) = \cos(x)$, n an integer, $\mathrm{g}(t) = 8\,\mathrm{sinc}\left[4\left(t - \frac{1}{4}\right)\right]\cos(40\pi t)$

(a) What is the maximum positive value of $\mathrm{g}(t)$ and at what time, t, does it occur?

The maximum value occurs at time, $t = \frac{1}{4}$, and the function value at that time is 8.

(b) What is the minimum positive value of time, $t > 0$, at which $\mathrm{g}(t)$ is zero?

The first zero crossing occurs when the $\cos(40\pi t)$ factor goes to zero for the first time after $t = 0$. That occurs at t = 1/80 s or 12.5 ms.

44. If $\mathrm{g}(x,y) = e^{-\pi(4x^2+9y^2)}$ find an expression for its two-dimensional Fourier transform, $\mathrm{G}(f_x, f_y)$.

$$\mathrm{g}(x,y) = e^{-\pi[4x^2+9y^2]} = e^{-\pi 4x^2}e^{-\pi 9y^2} = e^{-\pi(2x)^2}e^{-\pi(3y)^2}$$

Therefore this function is separable.

Therefore the two-dimensional Fourier transform is the product of two one- dimensional Fourier transforms, one for "x" and one for "y".

Using $e^{-\pi t^2} \xleftrightarrow{F} e^{-\pi f^2}$ and $g(at) \xleftrightarrow{F} \frac{1}{|a|} G\left(\frac{f}{a}\right)$,

and the fact that $g(x,y) = e^{-\pi(2x)^2} e^{-\pi(3y)^2}$

$$G(f_x, f_y) = \frac{1}{6} e^{-\pi\left(\frac{f_x}{2}\right)^2} e^{-\pi\left(\frac{f_y}{3}\right)^2} = \frac{1}{6} e^{-\pi\left(\frac{f_x^2}{4}+\frac{f_y^2}{9}\right)} .$$

(a) What is the maximum value of $\left|G(f_x, f_y)\right|$ and at what values of f_x and f_y does it occur?

The maximum value occurs where the exponent of *e* is at its most positive possible value. That occurs at $f_x = 0$ and $f_y = 0$ and the value there is $\frac{1}{6}$.

(b) What is the phase of $G(f_x, f_y)$ at $f_x = 1$ and $f_y = 1$? Since the function is purely real and positive the phase everywhere is zero.

45. Using contour integration in the complex plane and the method of residues find the numerical value of the definite integral,

$$I = \int_0^\infty \frac{dx}{x^4+1} .$$

Since the integrand is an even function,

$$I = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{x^4+1} .$$

Let this integral be part of the countor, C, which includes the real axis and an infinite-radius semicircle closed in the upper half plane. Then

$$I = \frac{1}{2} \int_{C,ccw} \frac{dz}{z^4+1}$$

$$I = \frac{1}{2} \times j2\pi \times \sum \text{residues at poles inside the contour}$$

The poles of $\frac{1}{z^4+1}$ lie at the roots of the equation, $z^4 + 1 = 0$ or at

$$z = e^{j\frac{\pi+2n\pi}{4}} = e^{j\pi\frac{2n+1}{4}} , \ n \text{ an integer}$$

The four distinct poles are

$$z = e^{j\frac{\pi}{4}}, e^{j\frac{3\pi}{4}}, e^{j\frac{5\pi}{4}}, e^{j\frac{7\pi}{4}}$$

and two of these are inside the contour, $z = e^{j\frac{\pi}{4}}, e^{j\frac{3\pi}{4}}$. The integrand can be factored into

$$\frac{1}{z^4+1} = \frac{1}{\left(z - e^{j\frac{\pi}{4}}\right)\left(z - e^{j\frac{3\pi}{4}}\right)\left(z - e^{j\frac{5\pi}{4}}\right)\left(z - e^{j\frac{7\pi}{4}}\right)}$$

Therefore

$$I = \frac{1}{2} \times j2\pi \times \left[\frac{1}{\left(e^{j\frac{\pi}{4}} - e^{j\frac{3\pi}{4}}\right)\left(e^{j\frac{\pi}{4}} - e^{j\frac{5\pi}{4}}\right)\left(e^{j\frac{\pi}{4}} - e^{j\frac{7\pi}{4}}\right)} + \frac{1}{\left(e^{j\frac{3\pi}{4}} - e^{j\frac{\pi}{4}}\right)\left(e^{j\frac{3\pi}{4}} - e^{j\frac{5\pi}{4}}\right)\left(e^{j\frac{3\pi}{4}} - e^{j\frac{7\pi}{4}}\right)} \right]$$

$$I = \frac{1}{2} \times j2\pi \times \left[\frac{1}{\left(e^{j\frac{\pi}{2}} - e^{j\pi} - e^{j\frac{3\pi}{2}} + e^{j2\pi}\right)\left(e^{j\frac{\pi}{4}} - e^{j\frac{7\pi}{4}}\right)} + \frac{1}{\left(e^{j\frac{3\pi}{2}} - e^{j\pi} - e^{j2\pi} + e^{j\frac{3\pi}{2}}\right)\left(e^{j\frac{3\pi}{4}} - e^{j\frac{7\pi}{4}}\right)} \right]$$

$$I = \frac{1}{2} \times j2\pi \times \left[\frac{1}{(j2+2)\left(\frac{1+j}{\sqrt{2}} - \frac{1-j}{\sqrt{2}}\right)} + \frac{1}{-j2\left(\frac{-1+j}{\sqrt{2}} - \frac{1-j}{\sqrt{2}}\right)} \right]$$

$$I = \frac{1}{2} \times j2\pi \times \left[\frac{1}{(j2+2)\frac{j2}{\sqrt{2}}} + \frac{1}{-j2\frac{-2+j2}{\sqrt{2}}} \right]$$

$$I = j\pi \times \left(\frac{\sqrt{2}}{-4+j4} + \frac{\sqrt{2}}{4+j4} \right) = j\pi \times \frac{\sqrt{2}(4+j4) + \sqrt{2}(-4+j4)}{(-4+j4)(4+j4)}$$

$$I = j\pi \times \frac{j8\sqrt{2}}{-32} = \frac{\sqrt{2}}{4}\pi \cong 1.111$$

Alternate Solution to #1:

Let this integral be part of the countor, C, which includes the positive real axis (C_1), an infinite-radius semicircle closed in the first quadrant, (C_2), and the positive imaginary axis, (C_3) traversed in the counter-clockwise sense. Then

$$j2\pi \times \sum \text{Residues at pole(s) in first quadrant} = \int_{C_1} \frac{\mathrm{dz}}{\mathrm{z}^4+1} + \int_{C_2} \frac{\mathrm{dz}}{\mathrm{z}^4+1} + \int_{C_3} \frac{\mathrm{dz}}{\mathrm{z}^4+1}$$

The integral value along C_2 is zero by Jordan's lemma.

On C_3, observe that

$$z = jx \text{ and } dz = jdx$$

Therefore

$$j2\pi \times \sum \text{Residues at pole(s) in first quadrant} = \int_{C_1} \frac{\mathrm{dz}}{\mathrm{z}^4+1} + \int_{C_3} \frac{\mathrm{dz}}{\mathrm{z}^4+1} = \int_0^\infty \frac{dx}{x^4+1} + \int_\infty^0 \frac{jdx}{(jx)^4+1}$$

$$j2\pi \times \sum \text{Residues at pole(s) in first quadrant} = \int_0^\infty \frac{dx}{x^4+1} - j\int_0^\infty \frac{dx}{x^4+1} = (1-j)I$$

The poles of $\frac{1}{z^4+1}$ lie at the roots of the equation, $z^4+1=0$ or at

$$z = e^{j\frac{\pi+2n\pi}{4}} = e^{j\pi\frac{2n+1}{4}} \text{ , } n \text{ an integer}$$

The four distinct poles are

$$z = e^{j\frac{\pi}{4}}, e^{j\frac{3\pi}{4}}, e^{j\frac{5\pi}{4}}, e^{j\frac{7\pi}{4}}$$

and one of these is inside the contour, $z = e^{j\frac{\pi}{4}}$. The integrand can be factored into

$$\frac{1}{z^4+1} = \frac{1}{\left(z-e^{j\frac{\pi}{4}}\right)\left(z-e^{j\frac{3\pi}{4}}\right)\left(z-e^{j\frac{5\pi}{4}}\right)\left(z-e^{j\frac{7\pi}{4}}\right)}$$

Therefore

$$(1-j)I = \frac{j2\pi}{\left(e^{j\frac{\pi}{4}}-e^{j\frac{3\pi}{4}}\right)\left(e^{j\frac{\pi}{4}}-e^{j\frac{5\pi}{4}}\right)\left(e^{j\frac{\pi}{4}}-e^{j\frac{7\pi}{4}}\right)}$$

$$I = \frac{j2\pi e^{j\frac{\pi}{4}}}{\sqrt{2}\left(e^{j\frac{\pi}{4}}-e^{j\frac{3\pi}{4}}\right)\left(e^{j\frac{\pi}{4}}-e^{j\frac{5\pi}{4}}\right)\left(e^{j\frac{\pi}{4}}-e^{j\frac{7\pi}{4}}\right)}$$

$$I = \frac{j2\pi\left(\frac{1+j}{\sqrt{2}}\right)}{\sqrt{2}(j+j+1+1)\left(e^{j\frac{\pi}{4}} - e^{j\frac{7\pi}{4}}\right)} = \frac{j\pi}{2\left(\frac{1+j}{\sqrt{2}} - \frac{1-j}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{4}\pi \cong 1.111$$

46. If the function, $x(t) = 3\,\text{rect}\left(\frac{t-2}{5}\right)$ is Fourier transformed the result is $X(f) = 15\,\text{sinc}(5f)e^{-j4\pi f}$ and it follows that $X\left(\frac{3}{8}\right) \cong -j0.9745$, or, stated more compactly,

$$F\left[3\,\text{rect}\left(\frac{t-2}{5}\right)\right]_{f\to\frac{3}{8}} = \left[15\,\text{sinc}(5f)e^{-j4\pi f}\right]_{f\to\frac{3}{8}} \cong -j0.9745 \quad .$$

In a similar manner find the Fourier transform and the numerical value for

$$F\left[\cos(8\pi t)\,\text{sinc}\left(\frac{t-1}{8}\right)\right]_{f\to 4.05}$$

$$F\left[\cos(8\pi t)\,\text{sinc}\left(\frac{t-1}{8}\right)\right] = \frac{1}{2}\left[\delta(f-4) + \delta(f+4)\right] * 8\,\text{rect}(8f)e^{-j2\pi f}$$

$$F\left[\cos(8\pi t)\,\text{sinc}\left(\frac{t-1}{8}\right)\right] = 4\left\{\text{rect}\left[8(f-4)\right]e^{-j2\pi(f-4)} + \text{rect}\left[8(f+4)\right]e^{-j2\pi(f+4)}\right\}$$

$$F\left[\cos(8\pi t)\,\text{sinc}\left(\frac{t-1}{8}\right)\right]_{f\to 4.05} = 4\left\{\text{rect}\left[8(0.05)\right]e^{-j2\pi(0.05)} + \text{rect}\left[8(8.05)\right]e^{-j2\pi(8.05)}\right\}$$

$$F\left[\cos(8\pi t)\,\text{sinc}\left(\frac{t-1}{8}\right)\right]_{f\to 4.05} = 4e^{-j2\pi(0.05)} = 4e^{-j0.314} = 4\angle -18^\circ = 3.804 - j1.236$$

47. A space-domain, two-dimensional function is given by

$$g(x,y) = \frac{1}{2}\left[1 + \cos(1600\pi x)\right]\text{rect}(100x)\,\text{rect}(20y) \quad .$$

Find the two-dimensional Fourier transform of this function, $G(f_x, f_y)$ and sketch the magnitudes of the cross sections of $G(f_x, f_y)$ along the f_x and along the f_y axis. That is sketch $\left|G(f_x,0)\right|$ and $\left|G(0,f_y)\right|$. (The sketches should include a scale and indication of the values of the independent and dependent variables at a few key points like maxima and minima and zero crossings so that rough estimates of the value of the function at other points could be made from the sketch. See example below.)

$$g(x,y) = \frac{1}{2}\left[\mathrm{rect}(100x) + \mathrm{rect}(100x)\cos(1600\pi x)\right]\mathrm{rect}(20y)$$

$$G(f_x, f_y) = \frac{1}{2}\left\{\frac{1}{100}\mathrm{sinc}\left(\frac{f_x}{100}\right) + \frac{1}{100}\mathrm{sinc}\left(\frac{f_x}{100}\right) * \frac{1}{2}\left[\delta(f_x - 800) + \delta(f_x + 800)\right]\right\}\frac{1}{20}\mathrm{sinc}\left(\frac{f_y}{20}\right)$$

$$G(f_x, f_y) = \frac{1}{4000}\left\{\mathrm{sinc}\left(\frac{f_x}{100}\right) + \frac{1}{2}\left[\mathrm{sinc}\left(\frac{f_x - 800}{100}\right) + \mathrm{sinc}\left(\frac{f_x + 800}{100}\right)\right]\right\}\mathrm{sinc}\left(\frac{f_y}{20}\right)$$

$$G(f_x, 0) = \frac{1}{4000}\left\{\mathrm{sinc}\left(\frac{f_x}{100}\right) + \frac{1}{2}\left[\mathrm{sinc}\left(\frac{f_x - 800}{100}\right) + \mathrm{sinc}\left(\frac{f_x + 800}{100}\right)\right]\right\}$$

$$G(0, f_y) = \frac{1}{4000}\mathrm{sinc}\left(\frac{f_y}{20}\right)$$