Solution of EE 503 Test #1 Su03 6/6/03 #1

1. (3 pts) If $z_1 = 1 + j$ and $z_2 = -3 - j$ find the numerical values of the magnitude and angle of $z_1 z_2$.

$$z_1 = \sqrt{2} \angle \frac{\pi}{4} \quad , \; z_2 = \sqrt{10} \angle -2.8198$$

$$z = 4.4721 \angle -2.0344 \pm 2n\pi$$

The only significant errors on this problem were that a couple of people got the angle in the wrong quadrant.

2. (7 pts) If one cube-root of a complex number is $j3$, find the numerical values of the real and imaginary parts of the other two.

The other two cube roots have the same magnitude, 3, and are at the angles, $\frac{\pi}{2} \pm \frac{2\pi}{3}$ because adjacent cube roots of any number are separated by an angle of $\frac{2\pi}{3}$.

Therefore the numerical values are $3 \angle \frac{7\pi}{6} = -2.5981 - j1.5$ and $3 \angle -\frac{\pi}{6} = 2.5981 - j1.5$.

I asked for numerical values of the real and imaginary parts. Some answers were written in the form, $\frac{3\sqrt{3}}{2}$. This is not a numerical value. It is a product of two numerical values divided by another numerical value. Those who gave the answer in this form got a small deduction.

3. (9 pts) Given a function, $w = z^3$, ($z = x + jy$, $w = u + jv$) graph the mapping of the contour, $x = y$, into the w plane.

$$w = z^3 = (x + jy)^3 = (x^2 + j2xy - y^2)(x + jy) = x^3 + j3x^2y - 3xy^2 - jy^3$$

$$u = x^3 - 3xy^2 \quad , \quad v = 3x^2y - y^3$$

If $x = y$,

$$u = -2x^3 \quad , \quad v = 2x^3 \Rightarrow u = -v$$

Therefore the contour in the w plane is a 45° line through the origin with a slope of negative one.

4. (8 pts) Is the function, $w = \exp\left(z^*\right)$, analytic in the entire z plane ? If not, where is it analytic and where is it not analytic?

$$u = e^x \cos(y) \ , \ v = -e^x \sin(y)$$

Cauchy-Riemann Conditions:

$$\frac{\partial \mathrm{u}}{\partial x} = +\frac{\partial \mathrm{v}}{\partial y}$$

$$\frac{\partial \mathrm{v}}{\partial x} = -\frac{\partial \mathrm{u}}{\partial y}$$

$$\frac{\partial u}{\partial x} = e^x \cos(y) \ , \ \frac{\partial v}{\partial y} = -e^x \cos(y)$$

$$\frac{\partial v}{\partial x} = -e^x \sin(y) \ , \ -\frac{\partial u}{\partial y} = e^x \sin(y)$$

It is not analytic anywhere in the z plane.

5. (14 pts) Find all the roots of $\sin(z) = -j$.

$$\frac{e^{jz} - e^{-jz}}{j2} = -j \Rightarrow \frac{e^{jz} - e^{-jz}}{2} = 1$$

$$\frac{e^{j(x+jy)} - e^{-j(x+jy)}}{2} = 1$$

$$\frac{e^{jx-y} - e^{-jx+y}}{2} = 1 \Rightarrow e^{-y}\left[\cos(x) + j\sin(x)\right] - e^{y}\left[\cos(x) - j\sin(x)\right] = 2$$

$$\left(e^{-y} - e^{y}\right)\cos(x) = 2 \ , \ \left(e^{-y} + e^{y}\right)\sin(x) = 0$$

$$\left(e^{-y} + e^{y}\right)\sin(x) = 0 \Rightarrow x = n\pi \quad , \quad n \text{ any integer}$$

$$x = n\pi \ \text{ and } \ \left(e^{-y} - e^{y}\right)\cos(x) = 2$$

For $y > 0$, $e^{-y} - e^{y} < 0$ and $x = \pi + 2n\pi$ and $e^{-y} - e^{y} = -2 \Rightarrow \sinh(y) = 1 \Rightarrow y = 0.8814$

For $y < 0$, $e^{-y} - e^{y} > 0$ and $x = 2n\pi$ and $\left(e^{-y} - e^{y}\right) = 2 \Rightarrow \sinh(y) = -1 \Rightarrow y = -0.8814$.

So the roots are $z = \pi + 2n\pi + j0.8814$ and $z = 2n\pi - j0.8814$.

Alternate Solution:

$$\sin(z) = \sin(x)\cosh(y) + j\cos(x)\sinh(y) = -j$$

$$\sin(x)\cosh(y) = 0 \;\;,\;\; \cos(x)\sinh(y) = -1$$

Since cosh is never zero, $\sin(x) = 0 \Rightarrow x = n\pi$, n an integer. That means that $\cos(x) = \pm 1$. When cos(x) is plus one, $x = 2n\pi$ and $\sinh(y) = -1 \Rightarrow y = \sinh^{-1}(-1) = -0.8814$. When cos($x$) is minus one, $x = 2n\pi + \pi$ and $\sinh(y) = 1 \Rightarrow y = \sinh^{-1}(1) = 0.8814$, and the roots are the same as found above.

6. (5 pts) Given the function, $w = \exp(z)$, map the region, $x < 0$, $3\pi < y < 4\pi$, into the w plane.

$$\exp(z) = e^x\left[\cos(y) + j\sin(y)\right]$$

If $x < 0$, then the magnitude of w is less than one.

If $3\pi < y < 4\pi$, then the angle of w lies in the same range, since y is the angle of w.

Therefore the region is the interior of the unit semicircle in the lower half of the w plane.

7. (5 pts) What are the residues of the function, $\dfrac{z}{z^2-4}$, at its two poles?

The poles are at $z = \pm 2$. The residues are $\left.\dfrac{z}{z+2}\right|_{z=2} = \dfrac{1}{2}$ and $\left.\dfrac{z}{z-2}\right|_{z=-2} = \dfrac{1}{2}$

8. (4 pts) On the branch, $-\pi \le \theta < \pi$, what is the numerical value of the logarithm of $-j4$?

$$\log(-j4) = \ln(4) + j\left(-\frac{\pi}{2}\right) = 1.3863 - j\frac{\pi}{2}$$

The angle of z must be $-\dfrac{\pi}{2}$ to be on the specified branch.