Solution of ECE 503 Test #3 Su03 6/20/03

1. (2 pts) Why is there no CTFS integration property for periodic signals with non-zero average values?

Because, if the average value is non-zero, the integral is not periodic and cannot, therefore, be represented for all time by a CTFS.

2. (10 pts) Let a signal be defined by

$$\mathrm{x}(t) = \mathrm{rect}(2t) * \mathrm{comb}(t) = \sum_{m=-\infty}^{\infty} \mathrm{rect}(2(t-m)) \ .$$

Using a representation time, $T_F = 2$, find the numerical values of the harmonic function, $\mathrm{X}[k]$, for $k = 0,1,2,3,4$.

Using the fundamental period of $\mathrm{x}(t)$ ($T_F = 1$) as the representation time, we get directly from the table,

$$\mathrm{rect}(2t) * \mathrm{comb}(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\mathrm{sinc}\left(\frac{k}{2}\right) \ .$$

Changing the period to $T_F = 2$ changes the harmonic function to

$$\mathrm{X}[k] = \begin{cases} \frac{1}{2}\mathrm{sinc}\left(\frac{k}{4}\right) & , \ \frac{k}{2} \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

Therefore,

$$\mathrm{X}[0] = \frac{1}{2} \ , \ \mathrm{X}[1] = 0 \ , \ \mathrm{X}[2] = \frac{1}{2}\mathrm{sinc}\left(\frac{1}{2}\right) = \frac{1}{\pi} \ , \ \mathrm{X}[3] = 0 \ , \ \mathrm{X}[4] = 0$$

Alternate Solution:

$$\mathrm{X}[k] = \frac{1}{T_F}\int_{T_F} \mathrm{x}(t) e^{-j2\pi(kf_F)t} dt = \frac{1}{2}\int_2 \underbrace{\mathrm{x}(t)}_{even}\left[\underbrace{\cos(\pi kt)}_{even} - \underbrace{j\sin(\pi kt)}_{odd}\right] dt$$

$$\mathrm{X}[k] = \frac{1}{2}\int_2 \mathrm{x}(t)\cos(\pi kt)\,dt = \frac{1}{2}\int_{-1}^{1} \mathrm{x}(t)\cos(\pi kt)\,dt = \int_0^1 \mathrm{x}(t)\cos(\pi kt)\,dt$$

For $k=0$,

$$\mathrm{X}[k]=\frac{1}{2}\int_{-1}^{1}\mathrm{x}(t)dt=\int_{0}^{1}\mathrm{x}(t)dt=\int_{0}^{\frac{1}{4}}dt+\int_{\frac{3}{4}}^{1}dt=\frac{1}{2}$$

For $k\neq 0$,

$$\mathrm{X}[k]=\int_{0}^{\frac{1}{4}}\cos(\pi kt)dt+\int_{\frac{3}{4}}^{1}\cos(\pi kt)dt=\left[\frac{\sin(\pi kt)}{\pi k}\right]_{0}^{\frac{1}{4}}+\left[\frac{\sin(\pi kt)}{\pi k}\right]_{\frac{3}{4}}^{1}=\frac{\sin\left(\frac{\pi k}{4}\right)-\sin\left(\frac{3\pi k}{4}\right)}{\pi k}$$

For any even *k* (except $k=0$), $\mathrm{X}[k]=0$.

For any odd *k*,

$$\sin\left(\frac{\pi k}{4}\right)=\sin\left(\frac{3\pi k}{4}\right)$$

and

$$\mathrm{X}[k]=\frac{2\sin\left(\frac{\pi k}{4}\right)}{\pi k}=\frac{1}{2}\mathrm{sinc}\left(\frac{k}{4}\right)$$

and these solutions agreee with the previous ones.

$$\mathrm{X}[0]=\frac{1}{2}\ ,\ \mathrm{X}[1]=0\ ,\ \mathrm{X}[2]=\frac{1}{2}\mathrm{sinc}\left(\frac{1}{2}\right)=\frac{1}{\pi}\ ,\ \mathrm{X}[3]=0\ ,\ \mathrm{X}[4]=0$$

3. (14 pts) Find the DTFS harmonic function of $\mathrm{x}[n]=2\cos\left(\frac{2\pi n}{8}\right)-5\sin\left(\frac{2\pi n}{12}\right)$ using the fundamental period of $\mathrm{x}[n]$ as the representation time, N_F.

$$N_F=24$$

Using the tables and the change-of-period property,

$$\mathrm{X}[k]=\left(\mathrm{comb}_{24}[k-3]+\mathrm{comb}_{24}[k+3]\right)-j\frac{5}{2}\left(\mathrm{comb}_{24}[k+2]-\mathrm{comb}_{24}[k-2]\right)\ .$$

4. A DT signal has a DTFT, $\mathrm{X}(F)=\left[\mathrm{rect}\left(5\left(F-\frac{1}{4}\right)\right)+\mathrm{rect}\left(5\left(F+\frac{1}{4}\right)\right)\right]*\mathrm{comb}(F)$.

(a) (8 pts) Find the signal's total signal energy, E_x.

$$E_x=\int_{1}\left|\left[\mathrm{rect}\left(5\left(F-\frac{1}{4}\right)\right)+\mathrm{rect}\left(5\left(F+\frac{1}{4}\right)\right)\right]*\mathrm{comb}(F)\right|^2 dF$$

$$E_x = \int_{\frac{1}{4}-\frac{1}{10}}^{\frac{1}{4}+\frac{1}{10}} dF + \int_{-\frac{1}{4}-\frac{1}{10}}^{-\frac{1}{4}+\frac{1}{10}} dF = \frac{2}{5}$$

(b) (7 pts) Find the signal, $x[n]$, and express it as a combination of real-valued DT functions (no *j*'s in the expression).

$$\text{sinc}\left(\frac{n}{w}\right) \xleftrightarrow{F} w\text{rect}(wF) * \text{comb}(F)$$

$$\text{sinc}\left(\frac{n}{5}\right) \xleftrightarrow{F} 5\text{rect}(5F) * \text{comb}(F)$$

$$\frac{1}{5}\text{sinc}\left(\frac{n}{5}\right) \xleftrightarrow{F} \text{rect}(5F) * \text{comb}(F)$$

$$\frac{1}{5}\text{sinc}\left(\frac{n}{5}\right)e^{+j\frac{\pi n}{2}} \xleftrightarrow{F} \text{rect}\left(5\left(F-\frac{1}{4}\right)\right) * \text{comb}(F)$$

$$\frac{1}{5}\text{sinc}\left(\frac{n}{5}\right)e^{-j\frac{\pi n}{2}} \xleftrightarrow{F} \text{rect}\left(5\left(F+\frac{1}{4}\right)\right) * \text{comb}(F)$$

$$\left(e^{+j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}\right)\frac{1}{5}\text{sinc}\left(\frac{n}{5}\right) \xleftrightarrow{F} \left[\text{rect}\left(5\left(F-\frac{1}{4}\right)\right) + \text{rect}\left(5\left(F+\frac{1}{4}\right)\right)\right] * \text{comb}(F)$$

$$\frac{2}{5}\cos\left(\frac{\pi n}{2}\right)\text{sinc}\left(\frac{n}{5}\right) \xleftrightarrow{F} \left[\text{rect}\left(5\left(F-\frac{1}{4}\right)\right) + \text{rect}\left(5\left(F+\frac{1}{4}\right)\right)\right] * \text{comb}(F)$$

5. (8 pts) Given the transform pair, $e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} e^{-\pi f^2}$, find the CTFT, $\mathrm{X}(f)$, of $\mathrm{x}(t) = e^{-\pi(3(t-2))^2}$. What is the numerical maximum value of $|\mathrm{X}(f)|$?

$$e^{-\pi(3t)^2} \xleftrightarrow{\mathcal{F}} \frac{1}{3} e^{-\pi\left(\frac{f}{3}\right)^2}$$

$$e^{-\pi(3(t-2))^2} \xleftrightarrow{\mathcal{F}} \frac{1}{3} e^{-\pi\left(\frac{f}{3}\right)^2} e^{-j4\pi f}$$

Maximum value occurs where $f = 0$. Value is $\frac{1}{3}$.

6. (2 pts) What is the fundamental reason that ideal filters cannot be actually built?

Their impulse responses are all infinite in extent and cannot, therefore, be the impulse responses of causal systems.