1. In the population of people world-wide about 10% of men are left-handed and about 6% of women are left-handed. In a randomly-chosen group of 60 women and 40 men what is the probability that at least one woman is left-handed and at least one man is left-handed?

$$\Pr\{\text{at least one woman in 60 is left-handed}\} = 1 - \Pr\{\text{none of the women is left-handed}\}$$

$$\Pr\{\text{none of the women is left-handed}\} = (0.94)^{60} = 0.02442$$

$$\therefore \Pr\{\text{at least one woman in 60 is left-handed}\} = 1 - 0.02442 = 0.97558$$

$$\Pr\{\text{at least one man in 40 is left-handed}\} = 1 - \Pr\{\text{none of the men is left-handed}\}$$

$$\Pr\{\text{none of the men is left-handed}\} = (0.9)^{40} = 0.01478$$

$$\therefore \Pr\{\text{at least one man in 40 is left-handed}\} = 1 - 0.01478 = 0.98522$$

$$\Pr\{\text{at least one woman left-handed and at least one man left-handed}\} = \Pr\{\text{at least one woman left-handed}\}\Pr\{\text{at least one man left-handed}\}$$

$$\Pr\{\text{at least one woman left-handed and at least one man left-handed}\} = 0.96116$$

2. A light source is randomly emitting photons which are striking a surface. The average rate at which photons strike the surface is $1000\frac{\text{photons}}{\text{sec}}$. If a photon arrives at time, t = 0, what is the probability that the next photon will arrive within 500 microseconds?

The mean time between photons is the reciprocal of the average rate of arrival.

$$\bar{T} = \frac{1}{1000\frac{\text{photons}}{\text{sec}}} = 0.001\frac{\text{sec}}{\text{photon}}$$

$$p_T(\tau) = \frac{e^{-\frac{\tau}{\bar{T}}}}{\bar{T}}\,\mathrm{u}(\tau)$$

$$\Pr\{\text{next photon within } 500\ \mu s\} = \int_{-\infty}^{500\mu s} p_T(\tau)\,d\tau = \int_{-\infty}^{500\mu s} \frac{e^{-\frac{\tau}{\bar{T}}}}{\bar{T}}\,\mathrm{u}(\tau)\,d\tau$$

$$\Pr\{\text{next photon within } 500\ \mu s\} = \int_{0}^{500\mu s} \frac{e^{-\frac{\tau}{\bar{T}}}}{\bar{T}}\,d\tau = \left[-e^{-\frac{\tau}{\bar{T}}}\right]_0^{500\mu s}$$

$$\Pr\left\{\text{next photon within } 500\ \mu s\right\} = -e^{-\frac{500\mu s}{1000\mu s}} + e^{-\frac{0}{1000\mu s}} = 1 - 0.6065 = 0.3935$$

3. Two independent random variables, X and Y, have probability density functions,

$$p_X(x) = \left\{\begin{matrix} \frac{1}{4} & , & -2 < X < 2 \\ 0 & , & \text{otherwise} \end{matrix}\right\} = \frac{1}{4}\text{rect}\left(\frac{X}{4}\right)$$

and

$$p_Y(y) = \left\{\begin{matrix} \frac{1}{2} & , & -1 < Y < 1 \\ 0 & , & \text{otherwise} \end{matrix}\right\} = \frac{1}{2}\text{rect}\left(\frac{Y}{2}\right) ,$$

If $Z = X + 2Y$, what is the probability that the magnitude of "Z" is greater than 2?

$$p_Z(z) = p_X(z) * p_{2Y}(z)$$

$$p_{2Y}(z) = \frac{1}{4}\text{rect}\left(\frac{z}{4}\right)$$

$$p_Z(z) = \frac{1}{4}\text{rect}\left(\frac{z}{4}\right) * \frac{1}{4}\text{rect}\left(\frac{z}{4}\right) = \frac{1}{4}\Lambda\left(\frac{z}{4}\right)$$

$$\Pr\left\{|Z| > 2\right\} = \Pr\left\{Z > 2\right\} + \Pr\left\{Z < -2\right\}$$

$$\Pr\left\{Z > 2\right\} = \int_2^4 \frac{1}{4}\Lambda\left(\frac{z}{4}\right)dz = \frac{1}{4}\int_2^4 \left(1 - \frac{z}{4}\right)dz = \frac{1}{4}\left[z - \frac{z^2}{8}\right]_2^4 = \frac{1}{8}$$

$$\Pr\left\{Z < -2\right\} = \Pr\left\{Z > 2\right\} = \frac{1}{8}$$

$$\Pr\left\{|Z| > 2\right\} = \frac{1}{4}$$

4. The correlation coefficient between two random variables, X and Y, is 0.5. The random variables, X and Y, have the same expected values and the same variances. That is,

$$\text{E}(X) = \text{E}(Y) \text{ and } \sigma_X^2 = \sigma_Y^2 \ .$$

A third random variable, Z is defined by

$$Z = 10X - 5Y \ .$$

What is the correlation coefficient between *X* and *Z*?

$$\rho_{XZ} = \frac{C_{XZ}}{\sigma_X \sigma_Z} = \frac{\mathrm{E}(XZ) - \mathrm{E}(X)\mathrm{E}(Z)}{\sigma_X \sigma_Z}$$

$$\mathrm{E}(Z) = 10\mathrm{E}(X) - 5\mathrm{E}(Y) = 5\mathrm{E}(X)$$

$$\rho_{XZ} = \frac{\mathrm{E}(XZ) - 5\left[\mathrm{E}(X)\right]^2}{\sigma_X \sigma_Z}$$

Using

$$\sigma_Z^2 = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_{X_i} \sigma_{X_j}$$

we get

$$\sigma_Z^2 = (10)^2 \sigma_X^2 + (-5)^2 \sigma_Y^2 + 2(10)(-5)\rho_{XY}\sigma_X\sigma_Y = 125\sigma_X^2 - 50\sigma_X^2 = 75\sigma_X^2$$

$$\Rightarrow \sigma_Z = \sqrt{75}\,\sigma_X \quad \text{and} \quad \rho_{XZ} = \frac{\mathrm{E}(XZ) - 5\left[\mathrm{E}(X)\right]^2}{\sqrt{75}\,\sigma_X^2}$$

$$\mathrm{E}(XZ) = \mathrm{E}\left[X(10X - 5Y)\right] = \mathrm{E}\left(10X^2 - 5XY\right) = 10\mathrm{E}\left(X^2\right) - 5\mathrm{E}(XY)$$

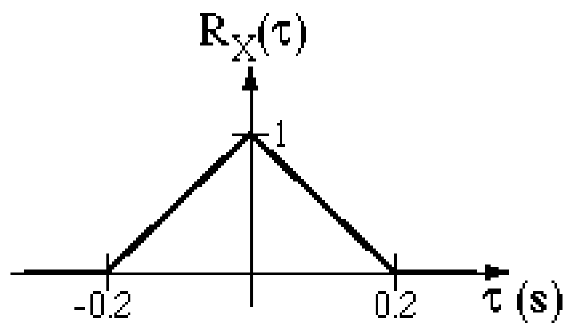
$$\mathrm{E}(XY) = \rho_{XY}\sigma_X\sigma_Y + \mathrm{E}(X)\mathrm{E}(Y) = 0.5\sigma_X^2 + \left[\mathrm{E}(X)\right]^2$$

$$\mathrm{E}(XZ) = 10\mathrm{E}\left(X^2\right) - 5\left\{0.5\sigma_X^2 + \left[\mathrm{E}(X)\right]^2\right\} = 5\mathrm{E}\left(X^2\right) - 2.5\sigma_X^2$$

$$\rho_{XZ} = \frac{10\mathrm{E}\left(X^2\right) - 2.5\sigma_X^2 - 10\left[\mathrm{E}(X)\right]^2}{\sqrt{75}\,\sigma_X^2} = \frac{10\sigma_X^2 - 2.5\sigma_X^2}{\sqrt{75}\,\sigma_X^2} = 0.866$$

5. An ergodic, continuous random process, {X(t)}, has the autocorrelation function, $R_X(\tau)$, graphed below. A sample function, X(t), of that random process is sampled at a rate, $f_s = \frac{1}{T_s}$. A discrete random variable is formed according to the following formula:

$$Y_0 = \frac{X(0) + X(T_s) + X(2T_s) + X(3T_s) + X(4T_s)}{5}$$
$$Y_1 = \frac{X(5T_s) + X(6T_s) + X(7T_s) + X(8T_s) + X(9T_s)}{5}$$
$$\vdots$$
$$Y_k = \frac{X(5kT_s) + X[(5k+1)T_s] + X[(5k+2)T_s] + +X[(5k+3)T_s] + +X[(5k+4)T_s]}{5}$$
$$\vdots$$



(a) If the sampling rate, fs, is 5 Hz, what is the variance of the random variable, Y ?

At a sampling rate of 5 Hz, the samples are uncorrelated because the autocorrelation function is zero for any Ts greater than or equal to 0.2 seconds. Therefore, the variance of the discrete random variable is simply,

$$\sigma_Y^2 = \sum_{i=0}^{4} \left(\frac{1}{5}\right)^2 \sigma_{Xi}^2 = \frac{1}{25}\sum_{i=0}^{4} \sigma_X^2 = \frac{\sigma_X^2}{5} = \frac{1}{5} = 0.2$$

(b) If the sampling rate, fs, is 10 Hz, what is the variance of the random variable, Y ?

In this case, there is some correlation between adjacent samples and the more general relationship between the variances of X and Y must be used. The expected value of X is zero, as can be seen from the autocorrelation function. Therefore the covariance between the two points, $X(iT_s)$ and $X(jT_s)$ is

$$\sigma_{X_iX_j} = \mathrm{E}(X_iX_j) - \mathrm{E}(X_i)\mathrm{E}(X_j) = \mathrm{E}(X_iX_j) = R_X[(j-i)T_s] \ .$$

and the correlation coefficient between X and Y is

$$\rho_{X_iX_j} = \frac{\sigma_{X_iX_j}}{\sigma_{X_i}\sigma_{X_j}} = \frac{R_X[(j-i)T_s]}{\sigma_X^2}$$

and

$$\sigma_Z^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \rho_{ij}\sigma_{X_i}\sigma_{X_j} = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} a_i a_j \rho_{ij}\sigma_{X_i}\sigma_{X_j}$$

$$\sigma_Y^2 = \sum_{i=0}^{4}\sum_{j=0}^{4} a_i a_j \rho_{ij}\sigma_{X_i}\sigma_{X_j} = \frac{1}{25}\sum_{i=0}^{4}\sum_{j=0}^{4}\frac{R_X\left[(j-i)T_s\right]}{\sigma_X^2}\sigma_X^2 = \frac{1}{25}\sum_{i=0}^{4}\sum_{j=0}^{4} R_X\left[(j-i)T_s\right]$$

$$\sigma_Y^2 = \frac{1}{25}\left[\begin{array}{l} R_X(0) + R_X(T_s) + R_X(2T_s) + R_X(3T_s) + R_X(4T_s) \\ + R_X(-T_s) + R_X(0) + R_X(T_s) + R_X(2T_s) + R_X(3T_s) \\ + R_X(-2T_s) + R_X(-T_s) + R_X(0) + R_X(T_s) + R_X(2T_s) \\ + R_X(-3T_s) + R_X(-2T_s) + R_X(-T_s) + R_X(0) + R_X(T_s) \\ + R_X(-4T_s) + R_X(-3T_s) + R_X(-2T_s) + R_X(-T_s) + R_X(0) \end{array}\right]$$

$$\sigma_Y^2 = \frac{1}{25}\left[\begin{array}{l} 1 + 0.5 + 0 + 0 + 0 \\ +0.5 + 1 + 0.5 + 0 + 0 \\ +0 + 0.5 + 1 + 0.5 + 0 \\ +0 + 0 + 0.5 + 1 + 0.5 \\ +0 + 0 + 0 + 0.5 + 1 \end{array}\right] = \frac{9}{25} = 0.36$$

6. White noise is the input to a filter whose impulse response is

$$h(t) = u(t) - u(t-4) = \text{rect}\left(\frac{t-2}{4}\right).$$

What is the maximum rate at which the output of the filter can be sampled and still have no correlation between samples?

$$H(f) = 4\,\text{sinc}(4f)e^{-j4\pi f}$$

Since the input noise is white, its PSD is of the form,

$$G_{in}(f) = A$$

where "A" is a constant. Therefore the PSD of the output would be

$$G_{out}(f) = 16A\,\text{sinc}^2(4f)$$

and the autocorrelation of the output would be

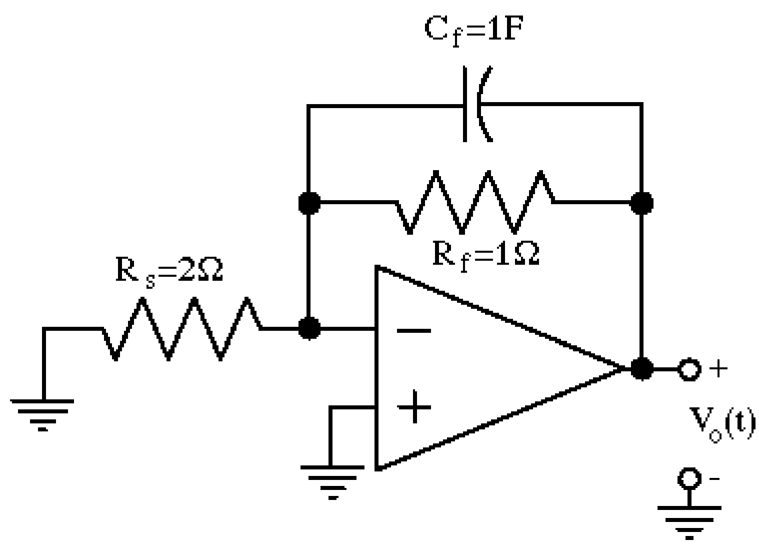
$$R_{out}(\tau) = 4A\Lambda\left(\frac{\tau}{4}\right).$$

For any sampling time less than the first zero of this autocorrelation there will be correlation between samples. That is for any sampling interval less than 4 seconds or any sampling rate greater than 0.25 Hz there will be correlation between samples.

7. In the circuit below the operational amplifier is ideal. That is, it has infinite gain, infinite input impedance, zero output impedance, zero noise and infinite bandwidth. This holds the voltage at the inverting input at zero volts. The two resistors are real and generate Johnson noise. The double-sided power spectral density of the equivalent noise current for the Johnson noise of a resistor is $\frac{2kT}{R}$ and the double-sided power spectral density of the equivalent noise voltage for the Johnson noise of a resistor is $2kTR$. The amplifier and resistors are all in thermal equilibrium at 300 K. The transfer function from the noise current of either resistor injected into the operational amplifier's inverting input to the output voltage is given by

$$H(f) = \frac{V_o(f)}{I(f)} = \square Z_f(f)$$

where Zf is the impedance of the feedback network of Rf in parallel with Cf. The noise voltages of the two resistors are independent. What is the mean-squared output noise voltage, $\mathrm{E}(V_o^2)$?



$$\mathrm{E}(V_o^2) = \underset{\square}{\square} G_o(f)\,df$$

$$G_o(f) = G_{os}(f) + G_{of}(s)$$

$$G_{os}(f) = G_s(f)|H(f)|^2 \quad \text{and} \quad G_{of}(f) = G_f(f)|H(f)|^2$$

$$G_o(f) = G_s(f)\left|H(f)\right|^2 + G_f(f)\left|H(f)\right|^2 = \left[G_s(f) + G_f(f)\right]\left|H(f)\right|^2$$

$$G_s(f) = \frac{2kT}{R_s} \quad \text{and} \quad G_f(f) = \frac{2kT}{R_f}$$

$$G_o(f) = 2kT\left(\frac{1}{R_s} + \frac{1}{R_f}\right)\left|H(f)\right|^2$$

$$Z_f(f) = \frac{\dfrac{R_f}{j2\pi fC}}{R_f + \dfrac{1}{j2\pi fC}} = \frac{R_f}{j2\pi fR_fC + 1}$$

$$G_o(f) = 2kT\left(\frac{1}{R_s} + \frac{1}{R_f}\right)\left|\frac{R_f}{j2\pi fR_fC + 1}\right|^2 = 2kT\left(\frac{1}{R_s} + \frac{1}{R_f}\right)\left|\frac{R_f}{j2\pi fR_fC + 1}\right|^2$$

$$G_o(f) = 2kT\left(\frac{1}{R_s} + \frac{1}{R_f}\right)\frac{R_f^2}{\left(2\pi fR_fC\right)^2 + 1}$$

$$\mathrm{E}\left(V_o^2\right) = 2kT\left(\frac{1}{R_s} + \frac{1}{R_f}\right)\int \frac{R_f^2}{\left(2\pi fR_fC\right)^2 + 1}\,df$$

$$\mathrm{E}\left(V_o^2\right) = \frac{2kT}{\left(2\pi R_fC\right)^2}\left(\frac{1}{R_s} + \frac{1}{R_f}\right)\int \frac{R_f^2}{f^2 + \dfrac{1}{\left(2\pi R_fC\right)^2}}\,df$$

$$\mathrm{E}\left(V_o^2\right) = \frac{2kTR_f^2}{\left(2\pi R_fC\right)^2}\left(\frac{1}{R_s} + \frac{1}{R_f}\right)\int \frac{df}{\left(f + j\dfrac{1}{2\pi R_fC}\right)\left(f - j\dfrac{1}{2\pi R_fC}\right)}$$

$$\mathrm{E}\left(V_o^2\right) = \frac{kT}{2\pi^2C^2}\left(\frac{1}{R_s} + \frac{1}{R_f}\right)j2\pi\left(\text{residue at } j\frac{1}{2\pi R_fC}\right)$$

$$\mathrm{E}\left(V_o^2\right) = \frac{kT}{C}\left(\frac{R_f}{R_s} + 1\right)$$

$$E\left(V_o^2\right) = \frac{3}{2}kT = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21}$$

8. A communication system transmits and receives binary messages (consisting only of binary bits, "0's" and "1's"). The probability that any randomly-chosen transmitted bit is a "1" is 0.45. Because of noise in the channel bits are sometimes received wrong. The probability that a trasmitted "1" will be received as a "0" is 0.05. The probability that a transmitted "0" will be received as a "1" is 0.1.

(a) Over a long period of time of operation of the system, what fraction of received bits are received correctly (the received bit is the same as the transmitted bit)?

$$\Pr(\text{correct received bit}) = \Pr(\text{sending and receiving a "1"}) + \Pr(\text{sending and receiving a "0"})$$

$$\Pr(\text{correct received bit}) = \Pr(\text{sending a "1"})\Pr(\text{sent "1" is received correctly}) + \Pr(\text{sending a "0"})\Pr(\text{sent "0" is received correctly})$$

$$\Pr(\text{correct received bit}) = 0.45 \times 0.95 + 0.55 \times 0.9 = 0.9225$$

(b) What is the probability of receiving 4 successive bits without error?

$$\Pr(\text{4 successive correct received bits}) = (0.9225)^4 = 0.7242$$

(c) What is the probability of receiving 4 successive bits with exactly one error?

$$\Pr(\text{1 error in 4 bits}) = \binom{4}{1} p^1 q^3 = 4 \times (1 - 0.9225)(0.9225)^3 = 0.2434$$

9. Two random variables, X and Y, have a joint probability density function given by

$$p_{XY}(x,y) = \begin{cases} Kxy, & 0 < x < 2 \text{ and } 0 < y < 2 \\ 0, & \text{elsewhere} \end{cases} = Kxy\,\mathrm{rect}\left(\frac{x-1}{2}\right)\mathrm{rect}\left(\frac{y-1}{2}\right)$$

(a) What is the numerical value of K?

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Kxy\,\mathrm{rect}\left(\frac{x-1}{2}\right)\mathrm{rect}\left(\frac{y-1}{2}\right)dxdy = K\int_0^2\int_0^2 xydxdy$$

$$1 = K\int_0^2\int_0^2 xydxdy = K\left[\int_0^2 xdx\right]\left[\int_0^2 ydy\right] = K\left[\frac{x^2}{2}\right]_0^2\left[\frac{y^2}{2}\right]_0^2 = 4K$$

$$\therefore K = \frac{1}{4}$$

(b) Find the marginal probability density functions, $p_X(x)$ and $p_Y(y)$.

$$p_X(x) = \int_{-\infty}^{\infty} Kxy\,\mathrm{rect}\left(\frac{x-1}{2}\right)\mathrm{rect}\left(\frac{y-1}{2}\right)dy = \frac{x}{4}\mathrm{rect}\left(\frac{x-1}{2}\right)\int_0^2 y\,dy = \frac{x}{2}\mathrm{rect}\left(\frac{x-1}{2}\right)$$

$$p_Y(y) = \int_{-\infty}^{\infty} Kxy\,\mathrm{rect}\left(\frac{x-1}{2}\right)\mathrm{rect}\left(\frac{y-1}{2}\right)dx = \frac{y}{4}\mathrm{rect}\left(\frac{y-1}{2}\right)\int_0^2 x\,dx = \frac{y}{2}\mathrm{rect}\left(\frac{y-1}{2}\right)$$

(c) Are X and Y statistically independent (uncorrelated)?

$$p_{XY}(x,y) = p_X(x)\,p_Y(y)$$

Therefore X and Y are statistically independent.

(d) If X is greater than 1.5, what is the probability that Y is less than 1? That is, find

$$\Pr\left(Y < 1 \text{ and } X > 1.5\right).$$

$$\Pr\left(Y < 1 \text{ and } X > 1.5\right) = \frac{1}{4}\int_{-\infty}^{1}\int_{1.5}^{\infty} xy\,\mathrm{rect}\left(\frac{x-1}{2}\right)\mathrm{rect}\left(\frac{y-1}{2}\right)dxdy$$

$$\Pr\left(Y < 1 \text{ and } X > 1.5\right) = \frac{1}{4}\int_0^1\int_{1.5}^2 xy\,dxdy = \frac{1}{4}\int_0^1 x\,dx\int_{1.5}^2 y\,dy = \frac{1}{4}\left(\frac{1}{2}\right)\left(2 - \frac{2.25}{2}\right) = 0.1094$$

(e) Find $E(XY)$.

$$E(XY) = \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2y^2\,\mathrm{rect}\left(\frac{x-1}{2}\right)\mathrm{rect}\left(\frac{y-1}{2}\right)dxdy = \frac{1}{4}\int_0^2\int_0^2 x^2y^2\,dxdy$$

$$E(XY) = \frac{1}{4}\int_0^2 x^2dx\int_0^2 y^2dy = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2\left[\frac{y^3}{3}\right]_0^2 = \frac{1}{4}\left(\frac{8}{3}\right)\left(\frac{8}{3}\right) = 1.778$$

10 Three zero-mean, unit-variance random variables, X, Y and Z are added to form a new random variable,

$$W = X + 2Y - Z \quad .$$

Random variables X and Y are uncorrelated, X and Z have a correlation coefficient of 0.3 and Y and Z have a correlation coefficient of -0.2. Find the variance of W.

$$\sigma_W^2 = \mathrm{E}(W^2) - [\mathrm{E}(W)]^2$$

$$\mathrm{E}(W) = \mathrm{E}(X + 2Y - Z) = \mathrm{E}(X) + 2\mathrm{E}(Y) - \mathrm{E}(Z) = 0$$

$$\Rightarrow \sigma_W^2 = \mathrm{E}(W^2)$$

$$\sigma_W^2 = \mathrm{E}(W^2) = \mathrm{E}\left[(X + 2Y - Z)^2\right] = \mathrm{E}\left(X^2 + 4Y^2 + Z^2 + 4XY - 2XZ - 4YZ\right)$$

$$\sigma_W^2 = \mathrm{E}\left(X^2 + 4Y^2 + Z^2 + 4XY - 2XZ - 4YZ\right)$$

$$\sigma_W^2 = \underbrace{\mathrm{E}(X^2)}_{\sigma_X^2=1} + 4\underbrace{\mathrm{E}(Y^2)}_{\sigma_Y^2=1} + \underbrace{\mathrm{E}(Z^2)}_{\sigma_Z^2=1} + 4\,\mathrm{E}(XY) - 2\mathrm{E}(XZ) - 4\,\mathrm{E}(YZ)$$

$$\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y) + \rho_{XY}\sigma_X\sigma_Y = 0$$

$$\mathrm{E}(XZ) = \mathrm{E}(X)\mathrm{E}(Z) + \rho_{XZ}\sigma_X\sigma_Z = 0.3(1)(1) = 0.3$$

$$\mathrm{E}(YZ) = \mathrm{E}(Y)\mathrm{E}(Z) + \rho_{YZ}\sigma_Y\sigma_Z = -0.2(1)(1) = -0.2$$

$$\sigma_W^2 = 1 + 4 + 1 - 2 \times 0.3 - 4 \times (-0.2) = 6 - 0.6 + 0.8 = 6.2$$

11. Two independent random variables, X and Y, both have a uniform probability density between -1 and 1 and zero probability density outside that range. What is the probability that a new random variable,

$$Z = 2X - 2Y \ ,$$

lies between 2 and 4?

$$p_X(x) = \frac{1}{2}\mathrm{rect}\left(\frac{x}{2}\right) \text{ and } p_Y(y) = \frac{1}{2}\mathrm{rect}\left(\frac{y}{2}\right)$$

Let

$$X' = 2X \text{ and } Y' = -2Y \ .$$

Then

$$p_{X'}(x) = \frac{1}{4}\mathrm{rect}\left(\frac{x}{4}\right) \text{ and } p_{Y'}(y) = \frac{1}{4}\mathrm{rect}\left(\frac{y}{4}\right)$$

and

$$p_Z(z) = p_{X'}(z) * p_{Y'}(z) = \frac{1}{4}\mathrm{rect}\left(\frac{z}{4}\right) * \frac{1}{4}\mathrm{rect}\left(\frac{z}{4}\right)$$

$$\Pr(2 < z < 4) = \int_2^4 p_Z(z)dz = \frac{1}{4}\int_2^4 \left(\frac{z}{4}\right)dz = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

12. In a warehouse there are 6 bins of resistors. Each bin contains resistors of various resistances according to the table below.

Resistance (Ω)	Bin 1	Bin 2	Bin 3	Bin 4	Bin 5	Bin 6	Total
10	500	0	200	800	1200	1000	3700
100	300	400	600	200	800	0	2300
1000	200	600	200	600	0	1000	2600
Total	1000	1000	1000	1600	2000	2000	8600

An experiment is carried out according to the rule that first a bin is selected randomly (meaning that each bin is equally likely to be selected) and then a resistor is drawn at random from that bin (meaning that each resistor in the bin is equally likely to be selected).

(a) What is the probability of selecting a 10 Ω resistor?

$$\Pr(10\Omega) = \Pr(10\Omega \mid \text{Bin 1})\Pr(\text{Bin 1}) + \Pr(10\Omega \mid \text{Bin 2})\Pr(\text{Bin 2}) + \Pr(10\Omega \mid \text{Bin 3})\Pr(\text{Bin 3})$$

$$+ \Pr(10\Omega \mid \text{Bin 4})\Pr(\text{Bin 4}) + \Pr(10\Omega \mid \text{Bin 5})\Pr(\text{Bin 5}) + \Pr(10\Omega \mid \text{Bin 6})\Pr(\text{Bin 6})$$

$$\Pr(10\Omega \mid \text{Bin 1}) = \frac{500}{1000} = \frac{1}{2}$$

$$\Pr(10\Omega \mid \text{Bin 2}) = \frac{0}{1000} = 0$$

$$\Pr(10\Omega \mid \text{Bin 3}) = \frac{200}{1000} = \frac{1}{5}$$

$$\Pr(10\Omega \mid \text{Bin 4}) = \frac{800}{1600} = \frac{1}{2}$$

$$\Pr(10\Omega \mid \text{Bin 5}) = \frac{1200}{2000} = \frac{3}{5}$$

$$\Pr(10\Omega \mid \text{Bin 6}) = \frac{1000}{2000} = \frac{1}{2}$$

$$\Pr(\text{Bin 1}) = \Pr(\text{Bin 2}) = \Pr(\text{Bin 3}) = \Pr(\text{Bin 4}) = \Pr(\text{Bin 5}) = \Pr(\text{Bin 6}) = \frac{1}{6}$$

$$\Pr(10\Omega) = \frac{1}{2}\times\frac{1}{6} + 0\times\frac{1}{6} + \frac{1}{5}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{6} + \frac{3}{5}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{6}$$

$$\Pr(10\Omega) = \frac{1}{12} + \frac{1}{30} + \frac{1}{12} + \frac{1}{10} + \frac{1}{12} = \frac{5+2+5+6+5}{60} = \frac{23}{60} = 0.38333\ldots$$

(b) If a 10 Ω resistor is selected, what is the probability that it came from Bin 1?

Using Bayes' theorem, $\Pr(A \mid B) = \dfrac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}$, $\Pr(B) > 0$,

$$\Pr\left(\text{Bin 1} \mid 10\Omega\right) = \frac{\Pr\left(10\Omega \mid \text{Bin 1}\right)\Pr\left(\text{Bin 1}\right)}{\Pr\left(10\Omega\right)} = \frac{\frac{1}{2}\times\frac{1}{6}}{\frac{23}{60}} = \frac{60}{276} \cong 0.2174$$

13. Two independent stationary random processes, X and Y, have autocorrelation functions given by

$$R_X(\tau) = 25e^{-10|\tau|}\cos(100\pi\tau) \quad , \quad R_Y(\tau) = 16\frac{\sin(50\pi\tau)}{50\pi\tau} = 16\,\mathrm{sinc}(50\tau) \quad .$$

Two other random variables are defined by

$$Z_s(t) = X(t) + Y(t) \text{ and } Z_d(t) = X(t) - Y(t) \quad .$$

Find

(a) The autocorrelation, $R_{Z_s}(\tau)$

$$R_{Z_s}(\tau) = \mathrm{E}\left[Z_s(t)Z_s(t+\tau)\right] = \mathrm{E}\left\{\left[X(t)+Y(t)\right]\left[X(t+\tau)+Y(t+\tau)\right]\right\}$$

$$R_{Z_s}(\tau) = \underbrace{\mathrm{E}\left[X(t)X(t+\tau)\right]}_{R_X(\tau)} + \underbrace{\mathrm{E}\left[X(t)Y(t+\tau)\right]}_{\mathrm{E}[X(t)]\mathrm{E}[Y(t)]} + \underbrace{\mathrm{E}\left[Y(t)X(t+\tau)\right]}_{\mathrm{E}[X(t)]\mathrm{E}[Y(t)]} + \underbrace{\mathrm{E}\left[Y(t)Y(t+\tau)\right]}_{R_Y(\tau)}$$

$$E\left[X(t)\right] = \pm\lim_{\tau\to\infty} 25e^{-10|\tau|}\cos(100\pi\tau) = 0$$
$$E\left[X(t)\right] = \pm\lim_{\tau\to\infty} 16\,\mathrm{sinc}(50\tau) = 0$$

$$R_{Z_s}(\tau) = R_X(\tau) + R_Y(\tau) = 25e^{-10|\tau|}\cos(100\pi\tau) + 16\,\mathrm{sinc}(50\tau)$$

and (b) the crosscorrelation, $R_{Z_sZ_d}(\tau)$.

$$R_{Z_sZ_d}(\tau) = \mathrm{E}\left\{Z_s(t)Z_d(t+\tau)\right\} = \mathrm{E}\left\{\left[X(t)+Y(t)\right]\left[X(t+\tau)-Y(t+\tau)\right]\right\}$$

$$R_{Z_sZ_d}(\tau) = \mathrm{E}\left[X(t)X(t+\tau) - Y(t)Y(t+\tau) - X(t)Y(t+\tau) + Y(t)X(t+\tau)\right]$$

$$R_{Z_sZ_d}(\tau) = \mathrm{E}\left[X(t)X(t+\tau)\right] - \mathrm{E}\left[Y(t)Y(t+\tau)\right] - \underbrace{\mathrm{E}\left[X(t)Y(t+\tau)\right]}_{=0} + \underbrace{\mathrm{E}\left[Y(t)X(t+\tau)\right]}_{=0}$$

$$R_{Z_sZ_d}(\tau) = R_X(\tau) - R_Y(\tau) = 25e^{-10|\tau|}\cos(100\pi\tau) - 16\,\mathrm{sinc}(50\tau)$$

14. A composite input signal, ri(t), consists of an input signal, si(t), plus input noise, ni(t). That is, $r_i(t) = s_i(t) + n_i(t)$. The input signal, si(t) is given by $s_i(t) = 100\sin(10{,}000\pi t)$. The input noise is white with a power spectral density of 0.1. The composite input signal, ri(t), is the input to a linear system with a transfer function of

$$H(f) = \frac{50}{1 + j\frac{f}{5000}} \quad .$$

The composite output signal, ro(t), consists of the sum of the response, so(t), to the input si(t) and the response, no(t), to the input ni(t). That is, $r_o(t) = s_o(t) + n_o(t)$. Find the ratio of the power in so(t) to the power in no(t). That is, find the signal-to-noise ratio,

$$SNR = \frac{signal\ power\ out}{noise\ power\ out} = \frac{P_s}{P_n} \quad .$$

$$S_i(f) = j\frac{100}{2}\left[\delta(f + 5000) - \delta(f - 5000)\right]$$

$$S_0(f) = S_i(f)H(f) = j\frac{100}{2}\left[\delta(f + 5000) - \delta(f - 5000)\right]\frac{50}{1 + j\frac{f}{5000}}$$

$$S_0(f) = j2500\frac{\delta(f + 5000) - \delta(f - 5000)}{1 + j\frac{f}{5000}} = j2500\left[\frac{\delta(f + 5000)}{1 - j} - \frac{\delta(f - 5000)}{1 + j}\right]$$

$$S_0(f) = j\frac{1 + j}{2}2500\delta(f + 5000) - j\frac{1 - j}{2}2500\delta(f - 5000)$$

$$S_0(f) = \frac{2500\sqrt{2}}{2}\left[e^{-j\frac{3\pi}{4}}\delta(f + 5000) + e^{-j\frac{5\pi}{4}}\delta(f - 5000)\right]$$

$$S_0(f) = -\frac{2500\sqrt{2}}{2}\left[\delta(f + 5000) + \delta(f - 5000)\right]e^{-j\frac{2\pi f}{40000}}$$

$$s_0(t) = -2500\sqrt{2}\cos\left(10{,}000\pi\left(t - \frac{1}{40000}\right)\right) = -2500\sqrt{2}\cos\left(10{,}000\pi t - \frac{\pi}{4}\right)$$

$$P_s = \frac{\left(-2500\sqrt{2}\right)^2}{2} = 2500^2$$

$$G_o(f) = G_i(f)\left|H(f)\right|^2 = 0.1\left|\frac{50}{1 + j\frac{f}{5000}}\right|^2 = \frac{250}{1 + \left(\frac{f}{5000}\right)^2}$$

$$P_n = \int_{-\infty}^{\infty} G_o(f)df = 250\int_{-\infty}^{\infty}\frac{df}{1 + \left(\frac{f}{5000}\right)^2}$$

Using

$$\int\frac{dx}{a^2 + (bx)^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)$$

$$P_n = 250\int_{-\infty}^{\infty}\frac{df}{1 + \left(\frac{f}{5000}\right)^2} = 250\left[5000\tan^{-1}\left(\frac{f}{5000}\right)\right]_{-\infty}^{\infty} = 1.25\times 10^6\pi$$

$$SNR = \frac{2500^2}{1.25 \times 10^6 \pi} \cong 1.5915$$

15. An ergodic random process, X, has a power spectral density given by

$$G_X(f) = 20\,\text{sinc}^2\left(\frac{f}{10}\right) .$$

(a) What is the mean-squared value of this random process?

$$\text{E}\left(X^2\right) = \int_{-\infty}^{\infty} G_X(f)\,df = 20\int_{-\infty}^{\infty} \text{sinc}^2\left(\frac{f}{10}\right) df$$

Using

$$g(0) = \int_{-\infty}^{\infty} G(f)\,df$$

$$\int_{-\infty}^{\infty} \text{sinc}^2\left(\frac{f}{10}\right) df = 10\,\Lambda(0) = 10$$

Therefore

$$\text{E}\left(X^2\right) = 20 \times 10 = 200$$

(b) What is the highest sampling rate at which samples from this process are statistically independent?

The autocorrelation function for this random process is

$$R_X(\tau) = 200\,\Lambda(10\tau)$$

Samples spaced closer than 0.1 seconds will be correlated. Therefore the highest sampling rate allowed is 10 Hz.

16. A random experiment is performed with a bowl containing marbles in random positions. The marbles are of various colors and consist of

5 black, 3 white, 5 red, 7 green, 9 blue, 1 yellow, 4 brown and 6 purple marbles.

The rules of the random experiment are to select a marble without looking, observe its color, replace it in the bowl and randomize the marble positions again. The outcome of the experiment is the color of the marble selected.

Find

(a) the probability that in three consecutive trials of the experiment the outcomes will all be the color, red

There are 40 marbles in the bowl and 5 of them are red.

$$\Pr(3 \text{ consecutive red marbles}) = \left(\frac{5}{40}\right)^3 = 0.001953$$

(b) the probability that in three consecutive trials of the experiment the outcomes will all be the same color

$$\Pr(3 \text{ same color}) = \left(\frac{5}{40}\right)^3 + \left(\frac{3}{40}\right)^3 + \left(\frac{5}{40}\right)^3 + \left(\frac{7}{40}\right)^3 + \left(\frac{9}{40}\right)^3 + \left(\frac{1}{40}\right)^3 + \left(\frac{4}{40}\right)^3 + \left(\frac{6}{40}\right)^3$$

$$\Pr(3 \text{ same color}) = 0.001953 + 0.000422 + 0.001953 + 0.005359 + 0.011391 + 0.000016 + 0.001000 + 0.003375$$

$$\Pr(3 \text{ same color}) = 0.025469$$

(c) the probability that in 10 trials of the experiment exactly 4 outcomes will be the color, black.

$$\Pr(4 \text{ blacks in any order in 10 trials}) = \binom{10}{4}\left(\frac{1}{8}\right)^4\left(\frac{7}{8}\right)^6 = 210 \times \frac{1}{4096} \times \frac{117649}{262144} = 0.023$$

17. A random variable, X, has a probability density function,

$$\mathrm{p}_X(x) = \begin{cases} \frac{1}{4} & , -3 < x < 1 \\ 0 & , \text{otherwise} \end{cases} = \frac{1}{4}\mathrm{rect}\left(\frac{x+1}{4}\right).$$

A random variable, Y, is related to X by

$$Y = X^2.$$

Find

(a) the expected value of Y, $\mathrm{E}(Y)$

$$\mathrm{E}(Y) = \mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 \mathrm{p}_X(x)\,dx = \frac{1}{4}\int_{-3}^{1} x^2\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_{-3}^{1} = \frac{7}{3} = 2.333\ldots$$

(b) the standard deviation of Y, σ_Y

$$\sigma_Y = \sqrt{\sigma_Y^2}$$

$$\sigma_Y^2 = \mathrm{E}\left(Y^2\right) - \left[\mathrm{E}(Y)\right]^2 = \mathrm{E}\left(X^4\right) - \left[\mathrm{E}(Y)\right]^2$$

$$\mathrm{E}\left(X^4\right) = \int_{-\infty}^{\infty} x^4\, \mathrm{p}_X(x)dx = \frac{1}{4}\int_{-3}^{1} x^4 dx = \frac{1}{4}\left[\frac{x^5}{5}\right]_{-3}^{1} = \frac{61}{5} = 12.2$$

$$\sigma_Y^2 = \frac{61}{5} - \left(\frac{7}{3}\right)^2 = \frac{61}{5} - \frac{49}{9} = \frac{549 - 245}{45} = \frac{304}{45} = 6.756$$

$$\sigma_Y = \sqrt{\frac{304}{45}} = 2.599$$

(c) the probability that any randomly-chosen Y will lie between 0 and 4, $\Pr(0 < Y < 4)$

$$\Pr(0 < Y < 4) = \Pr(0 < X < 2) + \Pr(-2 < X < 0)$$

$$\Pr(0 < X < 2) = \frac{1}{4} \text{ and } \Pr(-2 < X < 0) = \frac{1}{2}$$

$$\Pr(0 < Y < 4) = \frac{3}{4}$$

and

(d) the correlation coefficient between X and Y, ρ_{XY}.

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$\sigma_{XY} = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$$

$$\mathrm{E}(XY) = \mathrm{E}\left(X^3\right) = \int_{-\infty}^{\infty} x^3\, \mathrm{p}_X(x)dx = \frac{1}{4}\int_{-3}^{1} x^3 dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_{-3}^{1} = -5$$

$$\sigma_{XY} = -5 - (-1)\left(\frac{7}{3}\right) = -5 + \frac{7}{3} = -\frac{8}{3} = -2.666\ldots$$

$$\sigma_X = \sqrt{\sigma_X^2}$$

$$\sigma_X^2 = \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2 = \mathrm{E}(Y) - \left[\mathrm{E}(X)\right]^2 = \frac{7}{3} - (-1)^2 = \frac{4}{3} = 1.333\ldots$$

$$\sigma_X = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = 1.1547$$

$$\rho_{XY} = \frac{-\frac{8}{3}}{\frac{2}{\sqrt{3}} \sqrt{\frac{304}{45}}} = \frac{-\frac{8}{3}}{\sqrt{\frac{1216}{135}}} = -0.88852$$

18. Two independent random variables, X and Y, with probability density functions,

$$\mathrm{p}_X(x) = \begin{cases} 1 \ , \ 0 < x < 1 \\ 0 \ , \ \text{otherwise} \end{cases} = \mathrm{rect}\left(x - \frac{1}{2}\right)$$

and

$$\mathrm{p}_Y(y) = \begin{cases} \frac{1}{2} \ , \ -2 < y < 0 \\ 0 \ , \ \text{otherwise} \end{cases} = \frac{1}{2}\mathrm{rect}\left(\frac{y+1}{2}\right),$$

are combined to form a third random variable, $Z = 2X + Y$. What is the probability that any randomly-chosen Z will lie between -1 and 1, $\Pr(-1 < Z < 1)$?

$$\mathrm{p}_{2X}(x) = \begin{cases} \frac{1}{2} \ , \ 0 < x < 2 \\ 0 \ , \ \text{otherwise} \end{cases} = \frac{1}{2}\mathrm{rect}\left(\frac{x-1}{2}\right)$$

$$\mathrm{p}_Z(z) = \mathrm{p}_{2X}(z) * \mathrm{p}_Y(z) = \frac{1}{2}\mathrm{rect}\left(\frac{z+1}{2}\right) * \frac{1}{2}\mathrm{rect}\left(\frac{z-1}{2}\right)$$

$$\Pr(-1 < Z < 1) = \int_{-1}^{1} \mathrm{p}_Z(z)\,dz = \frac{1}{2}\int_{-1}^{1} \mathrm{tri}\left(\frac{z}{2}\right) dz = \int_{0}^{1} \mathrm{tri}\left(\frac{z}{2}\right) dz = \int_{0}^{1}\left(1 - \frac{z}{2}\right) dz$$

$$\Pr(-1 < Z < 1) = \left[z - \frac{z^2}{4}\right]_0^1 = \frac{3}{4}$$

19. A transmitted random signal, X, has a probability density function given by

$$\mathrm{p}_X(x) = \begin{cases} 2 - x & , \; 0 < x < 1 \\ 0 & , \; \text{otherwise} \end{cases} = 2(1 - x)\,\mathrm{rect}\left(x - \frac{1}{2}\right) .$$

The received signal, Y, is $X + N$ where N is random noise whose probability density function is given by

$$\mathrm{p}_N(n) = \begin{cases} \frac{1}{2} & , \; -1 < n < 1 \\ 0 & , \; \text{otherwise} \end{cases} = \frac{1}{2}\,\mathrm{rect}\left(\frac{n}{2}\right) .$$

X and N are statistically independent.

(a) Find an expression for the conditional probability density function, $\mathrm{p}_X(x \mid y)$.

$$\mathrm{p}_X(x \mid y)\,\mathrm{p}_Y(y) = \mathrm{p}_Y(y \mid x)\,\mathrm{p}_X(x)$$

$$\mathrm{p}_X(x \mid y) = \frac{\mathrm{p}_Y(y \mid x)\,\mathrm{p}_X(x)}{\mathrm{p}_Y(y)}$$

Then, using,

$$\mathrm{p}_X(x) = \int_{-\infty}^{\infty} \mathrm{p}_{XY}(x,y)\,dy = \int_{-\infty}^{\infty} \mathrm{p}_X(x \mid y)\,\mathrm{p}_Y(y)\,dy$$

$$\mathrm{p}_X(x \mid y) = \frac{\mathrm{p}_Y(y \mid x)\,\mathrm{p}_X(x)}{\int_{-\infty}^{\infty} \mathrm{p}_Y(y \mid x)\,\mathrm{p}_X(x)\,dx}$$

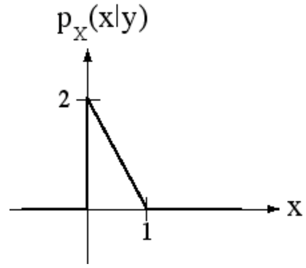
$$\mathrm{p}_Y(y \mid x) = \mathrm{p}_N(y - x)$$

$$\mathrm{p}_X(x \mid y) = \frac{\mathrm{p}_N(y - x)\,\mathrm{p}_X(x)}{\int_{-\infty}^{\infty} \mathrm{p}_N(y - x)\,\mathrm{p}_X(x)\,dx} = \frac{\frac{1}{2}\mathrm{rect}\left(\frac{y - x}{2}\right) 2(1 - x)\,\mathrm{rect}\left(x - \frac{1}{2}\right)}{\int_{-\infty}^{\infty} \frac{1}{2}\mathrm{rect}\left(\frac{y - x}{2}\right) 2(1 - x)\,\mathrm{rect}\left(x - \frac{1}{2}\right) dx}$$

$$\mathrm{p}_X(x \mid y) = \frac{\mathrm{rect}\left(\frac{y - x}{2}\right)(1 - x)\,\mathrm{rect}\left(x - \frac{1}{2}\right)}{\int_0^1 \mathrm{rect}\left(\frac{y - x}{2}\right)(1 - x)\,dx}$$

(b) Sketch $p_X(x \mid y)$ versus x given $Y = \frac{1}{2}$. The sketch should include scale information on both axes sufficient to allow calculation of numerical values of the function.

$$p_X\left(x \mid \frac{1}{2}\right) = \frac{\operatorname{rect}\left(\frac{\frac{1}{2} - x}{2}\right)(1 - x)\operatorname{rect}\left(x - \frac{1}{2}\right)}{\int_0^1 \operatorname{rect}\left(\frac{\frac{1}{2} - x}{2}\right)(1 - x)\,dx} = \frac{(1 - x)\operatorname{rect}\left(x - \frac{1}{2}\right)}{\int_0^1 (1 - x)\,dx} = 2(1 - x)\operatorname{rect}\left(x - \frac{1}{2}\right)$$

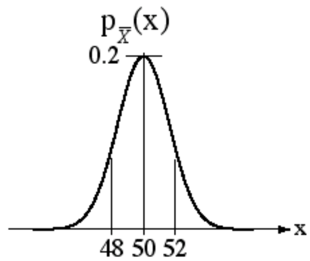


20. A random variable whose probability density function is Gaussian is sampled 100 times. One particular sample mean, $\bar{x}$, calculated from these 100 samples is 50 and one particular unbiased sample standard deviation, S_X, calculated from the same 100 samples is

(a) Given the information you have, sketch your best estimate of the probability density function for sample means, $\bar{X}$, calculated from 100 samples from this same population. The sketch should include scale information on both axes sufficient to allow calculation of numerical values of the function.

The best estimate of the variance of the sample mean is $\sigma_{\bar{X}}^2 = \frac{20^2}{100} = 4$. Therefore the best estimate of the standard deviation of the sample mean is $\sigma_{\bar{X}} = 2$. The best estimate of the population mean is the sample mean, 50. The distribution of the sample mean will be Gaussian with the probability density function,

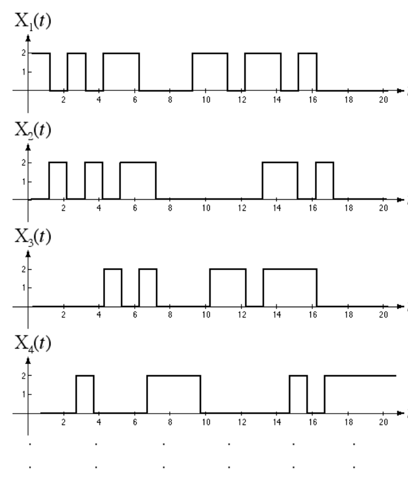
$$p_{\bar{X}}(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-50)^2}{8}} .$$



(b) What is the probability that the magnitude of the difference between this particular sample mean, $\bar{x}$, and the population mean is less than 0.5?

Looking at the table of the normal distribution the probability is about 19.7%.

21. An ergodic, deterministic random process has sample functions each of which is a sequence of contiguous rectangular pulses of width 1 second. The pulses can have only two possible amplitudes, 2 and 0, with equal probability. The pulse amplitudes occur randomly and are all statistically independent of each other. Sketch the autocorrelation function for this random process. The sketch should include scale information on both axes sufficient to allow calculation of numerical values of the function.



The definition of autocorrelation is

$$R_X(\tau) = E\left[X(t)\,X(t+\tau)\right] \; .$$

The autocorrelation is

$$R_X(\tau) = E[X(t)\,X(t+\tau)] = \begin{bmatrix} 2^2 \times \Pr(2) \times \Pr(t+\tau \text{ is in same pulse}) \\ +0^2 \times \Pr(0) \times \Pr(t+\tau \text{ is in same pulse}) \\ +2 \times 2 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(2 - 2) \\ +2 \times 0 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(2 - 0) \\ +0 \times 2 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(0 - 2) \\ +0 \times 0 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(0 - 0) \end{bmatrix}$$

$$R_X(\tau) = E[X(t)\,X(t+\tau)] = \begin{bmatrix} 4 \times \frac{1}{2} \times \text{tri}(\tau) + 0 \times \frac{1}{2} \times \text{tri}(\tau) + 4 \times [1 - \text{tri}(\tau)] \times \frac{1}{4} \\ +0 \times [1 - \text{tri}(\tau)] \times \frac{1}{4} + 0 \times [1 - \text{tri}(\tau)] \times \frac{1}{4} + 0 \times [1 - \text{tri}(\tau)] \times \frac{1}{4} \end{bmatrix}$$

$$R_X(\tau) = E[X(t)\,X(t+\tau)] = \text{tri}(\tau) + 1$$

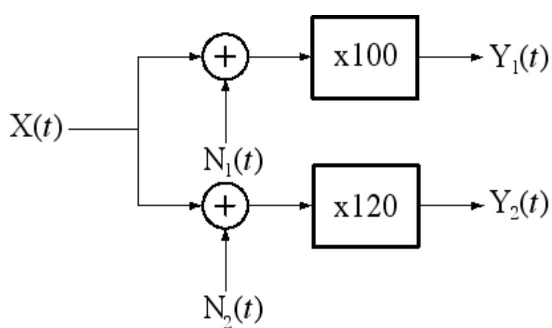
22. In the system below, all three inputs, X, N_1 and N_2, are bandlimited white noise with a bandwidth of 100 kHz. That is,

$$G_X(f) = G_{N_1}(f) = G_{N_2}(f) = 0 \quad , \quad |f| > 100 \text{ kHz} .$$

All three signals are mutually statistically independent. Their mean-squared values are

$$E(X) = 2 \quad , \quad E(N_1) = 1 \quad , \quad E(N_2) = 1 \quad .$$

X is added to N_1 and N_2 and then amplified by the two amplifiers with gains of 100 and 120, as indicated in the diagram. Sketch the cross power spectral density, $G_{Y_1 Y_2}(f)$, of the outputs, Y_1 and Y_2. The sketch should have scales on both axes which allow the reading of numerical values of the function.



$$R_{Y_1 Y_2}(\tau) = E[Y_1(t)\,Y_2(t+\tau)] = E\{100[X(t) + N_1(t)]120[X(t+\tau) + N_2(t+\tau)]\}$$

$$R_{Y_1 Y_2}(\tau) = 12000\,E[X(t)\,X(t+\tau) + X(t)N_2(t+\tau) + N_1(t)X(t+\tau) + N_1(t)N_2(t+\tau)]$$

$$R_{Y_1Y_2}(\tau) = 12000\,E\left[X(t)\,X(t+\tau)\right] = 12000\,R_X(\tau)$$

$$\Rightarrow G_{Y_1Y_2}(f) = 12000\,G_X(f) = 12000\,\frac{2}{200000}\,\mathrm{rect}\left(\frac{f}{200000}\right) = 0.12\,\mathrm{rect}\left(\frac{f}{200000}\right)$$ 24. In a

23. In a hardware store there are 3 bins containing bolts and 3 bins containing nuts as detailed in the tables below.

Bins A-C contain 8mm, 10mm and 12mm diameter bolts:

	Bin A	Bin B	Bin C
8mm	0	200	100
10mm	200	300	100
12mm	200	0	200

Bins D-F contain 8mm, 10mm and 12mm diameter nuts:

	Bin D	Bin E	Bin F
8mm	300	100	200
10mm	100	0	200
12mm	100	200	0

A customer first randomly picks a bin A-C and then randomly takes a bolt from that bin. Then the customer randomly picks a bin D-F and then randomly takes a nut from that bin.

What is the numerical probability that the bolt and the nut have the same diameter?

$$\Pr(\text{same size bolt and nut}) = \begin{bmatrix} \Pr(\text{8mm bolt and 8mm nut}) \\ +\Pr(\text{10mm bolt and 10mm nut}) \\ +\Pr(\text{12mm bolt and 12mm nut}) \end{bmatrix}$$

$$\Pr(\text{8mm bolt and 8mm nut}) = \Pr(\text{8mm bolt})\Pr(\text{8mm nut})$$

$$\Pr(\text{8mm bolt}) = \begin{bmatrix} \Pr(\text{BinA})\Pr(\text{8mm bolt}\mid\text{BinA}) \\ +\Pr(\text{BinB})\Pr(\text{8mm bolt}\mid\text{BinB}) \\ +\Pr(\text{BinC})\Pr(\text{8mm bolt}\mid\text{BinC}) \end{bmatrix} = \frac{1}{3}\left(\frac{0}{400} + \frac{200}{500} + \frac{100}{400}\right) = 0.217$$

$$\Pr(\text{8mm nut}) = \begin{bmatrix} \Pr(\text{BinD})\Pr(\text{8mm nut}\mid\text{BinD}) \\ +\Pr(\text{BinE})\Pr(\text{8mm nut}\mid\text{BinE}) \\ +\Pr(\text{BinF})\Pr(\text{8mm nut}\mid\text{BinF}) \end{bmatrix} = \frac{1}{3}\left(\frac{300}{500} + \frac{100}{300} + \frac{200}{400}\right) = 0.478$$

$$\Pr(\text{8mm bolt and 8mm nut}) = 0.217 \times 0.478 = 0.104$$

$$\Pr(\text{10mm bolt}) = \begin{bmatrix} \Pr(\text{BinA})\Pr(\text{10mm bolt} \mid \text{BinA}) \\ + \Pr(\text{BinB})\Pr(\text{10mm bolt} \mid \text{BinB}) \\ + \Pr(\text{BinC})\Pr(\text{10mm bolt} \mid \text{BinC}) \end{bmatrix} = \frac{1}{3}\left(\frac{200}{400} + \frac{300}{500} + \frac{100}{400}\right) = 0.45$$

$$\Pr(\text{10mm nut}) = \begin{bmatrix} \Pr(\text{BinD})\Pr(\text{10mm nut} \mid \text{BinD}) \\ + \Pr(\text{BinE})\Pr(\text{10mm nut} \mid \text{BinE}) \\ + \Pr(\text{BinF})\Pr(\text{10mm nut} \mid \text{BinF}) \end{bmatrix} = \frac{1}{3}\left(\frac{100}{500} + \frac{0}{300} + \frac{200}{400}\right) = 0.233$$

$$\Pr(\text{10mm bolt and 10mm nut}) = 0.45 \times 0.233 = 0.105$$

$$\Pr(\text{12mm bolt}) = \begin{bmatrix} \Pr(\text{BinA})\Pr(\text{12mm bolt} \mid \text{BinA}) \\ + \Pr(\text{BinB})\Pr(\text{12mm bolt} \mid \text{BinB}) \\ + \Pr(\text{BinC})\Pr(\text{12mm bolt} \mid \text{BinC}) \end{bmatrix} = \frac{1}{3}\left(\frac{200}{400} + \frac{0}{500} + \frac{200}{400}\right) = 0.333$$

$$\Pr(\text{12mm nut}) = \begin{bmatrix} \Pr(\text{BinD})\Pr(\text{12mm nut} \mid \text{BinD}) \\ + \Pr(\text{BinE})\Pr(\text{12mm nut} \mid \text{BinE}) \\ + \Pr(\text{BinF})\Pr(\text{12mm nut} \mid \text{BinF}) \end{bmatrix} = \frac{1}{3}\left(\frac{100}{500} + \frac{200}{300} + \frac{0}{400}\right) = 0.289$$

$$\Pr(\text{12mm bolt and 12mm nut}) = 0.333 \times 0.289 = 0.096$$

$$\Pr(\text{same size bolt and nut}) = 0.104 + 0.105 + 0.096 = 0.305$$

24. Two random variables, X and Y, have a joint distribution function,

$$\mathrm{F}_{XY}(x,y) = \frac{1}{2}\left[\mathrm{u}(x-2) + \mathrm{u}(x-3)\right]\left(1 - e^{-\frac{y}{2}}\right)\mathrm{u}(y)$$

Find

(a) The joint pdf, $\mathrm{p}_{XY}(x,y)$,

$$\mathrm{p}_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y}\left(\mathrm{F}_{XY}(x,y)\right) = \frac{\partial}{\partial y}\left\{\frac{1}{2}\left[\delta(x-2) + \delta(x-3)\right]\left(1 - e^{-\frac{y}{2}}\right)\mathrm{u}(y)\right\}$$

$$\mathrm{p}_{XY}(x,y) = \frac{1}{2}\left[\delta(x-2) + \delta(x-3)\right] \times \frac{1}{2} e^{-\frac{y}{2}}\,\mathrm{u}(y) = \frac{1}{4}\left[\delta(x-2) + \delta(x-3)\right] e^{-\frac{y}{2}}\,\mathrm{u}(y)$$

(b) The marginal pdf, $\mathrm{p}_X(x)$, and the marginal pdf, $\mathrm{p}_Y(y)$,

$$\mathrm{p}_X(x) = \int_{-\infty}^{\infty} \mathrm{p}_{XY}(x,y)\,dy = \int_{-\infty}^{\infty} \frac{1}{4}\left[\delta(x-2) + \delta(x-3)\right] e^{-\frac{y}{2}}\,\mathrm{u}(y)\,dy$$

$$\mathrm{p}_X(x) = \frac{\delta(x-2) + \delta(x-3)}{4} \int_0^{\infty} e^{-\frac{y}{2}}\,dy = \frac{\delta(x-2) + \delta(x-3)}{4}\left[-2e^{-\frac{y}{2}}\right]_0^{\infty}$$

$$\mathrm{p}_X(x) = \frac{\delta(x-2) + \delta(x-3)}{4}(0+2) = \frac{\delta(x-2) + \delta(x-3)}{2}$$

$$\mathrm{p}_Y(y) = \int_{-\infty}^{\infty} \mathrm{p}_{XY}(x,y)\,dx = \int_{-\infty}^{\infty} \frac{1}{4}\left[\delta(x-2) + \delta(x-3)\right] e^{-\frac{y}{2}}\,\mathrm{u}(y)\,dx$$

$$\mathrm{p}_Y(y) = \frac{e^{-\frac{y}{2}}}{4}\,\mathrm{u}(y) \int_{-\infty}^{\infty} \left[\delta(x-2) + \delta(x-3)\right] dx = \frac{e^{-\frac{y}{2}}}{4}\,\mathrm{u}(y)(2) = \frac{e^{-\frac{y}{2}}}{2}\,\mathrm{u}(y)$$

and (c) The numerical probability that X and Y are simultaneously between 1.5 and 3.5, $\Pr(1.5 < X \le 3.5 \text{ and } 1.5 < Y \le 3.5)$.

$$\Pr(1.5 < X \le 3.5 \text{ and } 1.5 < Y \le 3.5) = \int_{1.5}^{3.5}\int_{1.5}^{3.5} \mathrm{p}_{XY}(x,y)\,dx\,dy$$

$$\Pr(1.5 < X \le 3.5 \text{ and } 1.5 < Y \le 3.5) = \int_{1.5}^{3.5}\int_{1.5}^{3.5} \frac{1}{4}\left[\delta(x-2) + \delta(x-3)\right] e^{-\frac{y}{2}}\,\mathrm{u}(y)\,dx\,dy$$

$$\Pr(1.5 < X \le 3.5 \text{ and } 1.5 < Y \le 3.5) = \int_{1.5}^{3.5} e^{-\frac{y}{2}}\,\mathrm{u}(y)\,dy \int_{1.5}^{3.5} \frac{1}{4}\left[\delta(x-2) + \delta(x-3)\right] dx$$

$$\Pr(1.5 < X \le 3.5 \text{ and } 1.5 < Y \le 3.5) = \frac{1}{2}\int_{1.5}^{3.5} e^{-\frac{y}{2}}\,dy = \frac{1}{4}\left[-2e^{-\frac{y}{2}}\right]_{1.5}^{3.5} = \frac{1}{2}\left[-2e^{-\frac{3.5}{2}} + 2e^{-\frac{1.5}{2}}\right]$$

$$\Pr(1.5 < X \le 3.5 \text{ and } 1.5 < Y \le 3.5) = \left[-0.173 + 0.472\right] = 0.299$$

25. A random variable, X, has a pdf,

$$\mathrm{p}_X(x) = \begin{cases} \frac{1}{2} & , \ -2 < x < 0 \\ 0 & , \ \text{otherwise} \end{cases} = \frac{1}{2}\,\mathrm{rect}\left(\frac{x+1}{2}\right).$$

Two other random variables, Y and Z are related to X by

$$Y = -2X + 2 \quad , \quad Z = -3X \quad .$$

Find

(a) The expected values of *Y* and *Z*, $\mathrm{E}(Y)$ and $\mathrm{E}(Z)$,

$$\mathrm{E}(X) = -1$$

$$\mathrm{E}(Y) = \mathrm{E}(-2X + 2) = -2\mathrm{E}(X) + 2 = 2 + 2 = 4$$

$$\mathrm{E}(Z) = \mathrm{E}(-3X) = -3\mathrm{E}(X) = -3(-1) = 3$$

(b) The expected value of the product, *YZ*, $\mathrm{E}(YZ)$,

$$\mathrm{E}(YZ) = \mathrm{E}\left[(-2X + 2)(-3X)\right] = \mathrm{E}\left[6X^2 - 6X\right] = 6\mathrm{E}\left(X^2\right) - 6\mathrm{E}(X)$$

$$\mathrm{E}(YZ) = 6\mathrm{E}\left(X^2\right) - 6(-1) = 6\mathrm{E}\left(X^2\right) + 6$$

$$\mathrm{E}\left(X^2\right) = \int_{-\infty}^{\infty} \frac{x^2}{2} \mathrm{rect}\left(\frac{x+1}{2}\right) dx = \frac{1}{2}\int_{-2}^{0} x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{0} = \frac{4}{3}$$

$$\mathrm{E}(YZ) = 6\left(\frac{4}{3}\right) + 6 = 14$$

and (c) The variances of *Y* and *Z*, σ_Y^2 and σ_Z^2.

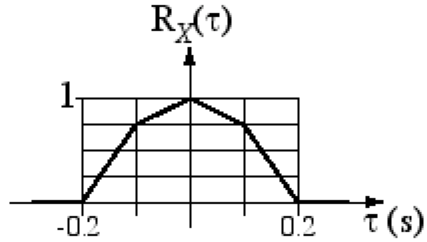
$$\sigma_Y^2 = \left(\frac{\partial Y}{\partial X}\right)^2 \sigma_X^2$$

$$\sigma_X^2 = \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2 = \frac{4}{3} - (-1)^2 = \frac{1}{3}$$

$$\sigma_Y^2 = (-2)^2 \frac{1}{3} = \frac{4}{3}$$

$$\sigma_Z^2 = \left(\frac{\partial Z}{\partial X}\right)^2 \sigma_X^2 = (-3)^2 \frac{1}{3} = 3$$

26. An ergodic, continuous random process, $\{X(t)\}$, has the autocorrelation function, $R_X(\tau)$, graphed below. A sample function, $X(t)$, of that random process is sampled at a rate, $f_s = \frac{1}{T_s}$. A discrete random variable is formed according to the following formula:



$$Y_0 = \frac{X(0) + 2X(T_s) + X(2T_s)}{4}$$

$$Y_1 = \frac{X(3T_s) + 2X(4T_s) + X(5T_s)}{4}$$

$$\vdots$$

$$Y_k = \frac{X(3kT_s) + 2X[(3k+1)T_s] + X[(3k+2)T_s]}{4}$$

$$\vdots$$

(a) If the sampling rate, f_s, is 5 Hz, what is the variance of the random variable, Y ?

At this sampling rate the samples are all independent. Therefore the variance is simply

$$\sigma_Y^2 = \left(\frac{1}{4}\right)^2 \sigma_X^2 + \left(\frac{2}{4}\right)^2 \sigma_X^2 + \left(\frac{1}{4}\right)^2 \sigma_X^2 = \frac{6}{16}\sigma_X^2 = \frac{3}{8}\sigma_X^2$$

From the autocorrelation graph, $\sigma_X^2 = 1$. Therefore $\sigma_Y^2 = \frac{3}{8}$.

(b) If the sampling rate, f_s, is 10 Hz, what is the variance of the random variable, Y ?

At this sampling rate adjacent samples are partially correlated. Therefore the variance is given by the general formula,

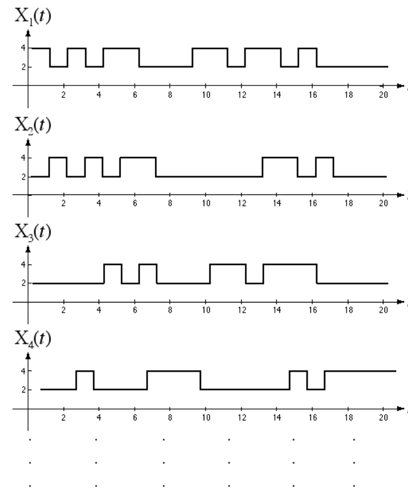
$$\sigma_Y^2 = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j \sigma_{X_i X_j} = \sum_{j=1}^{3}\sum_{i=1}^{3} a_i a_j \sigma_{X_i X_j} = \sum_{j=1}^{3}\sum_{i=1}^{3} a_i a_j C_X\left((i-j)T_s\right) = \sum_{j=1}^{3}\sum_{i=1}^{3} a_i a_j R_X\left((i-j)T_s\right)$$

From the autocorrelation function the covariances are

$$\sigma_{11} = 1 \quad \sigma_{12} = 0.75 \quad \sigma_{13} = 0$$
$$\sigma_{21} = 0.75 \quad \sigma_{22} = 1 \quad \sigma_{23} = 0.75$$
$$\sigma_{31} = 0 \quad \sigma_{32} = 0.75 \quad \sigma_{33} = 1$$

$$\sigma_Y^2 = \left(\frac{1}{4}\right)^2 \begin{bmatrix} 1 \times 1(1) + 1 \times 2(0.75) + 1 \times 1(0) \\ +2 \times 1(0.75) + 2 \times 2(1) + 2 \times 1(0.75) \\ +1 \times 1(0) + 1 \times 2(0.75) + 1 \times 1(1) \end{bmatrix} = \frac{1 + 1.5 + 1.5 + 4 + 1.5 + 1.5 + 1}{16} = \frac{3}{4}$$

27. An ergodic, deterministic random process has sample functions each of which is a sequence of contiguous rectangular pulses of width 1 second. The pulses can have only two possible amplitudes, 2 and 4, with equal probability. The pulse amplitudes occur randomly and are all statistically independent of each other. Sketch the autocorrelation function for this random process. The sketch should include scale information on both axes sufficient to allow calculation of numerical values of the function.



The definition of autocorrelation is

$$R_X(\tau) = E\left[X(t)\,X(t+\tau)\right] \ .$$

The autocorrelation is

$$R_X(\tau) = E\left[X(t)\,X(t+\tau)\right] = \left[\begin{array}{l} 4^2 \times \Pr(4) \times \Pr(t+\tau \text{ is in same pulse}) \\ +2^2 \times \Pr(2) \times \Pr(t+\tau \text{ is in same pulse}) \\ +4 \times 4 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(4 - 4) \\ +4 \times 2 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(4 - 2) \\ +2 \times 4 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(2 - 4) \\ +2 \times 2 \times \Pr(t+\tau \text{ is in different pulse}) \times \Pr(2 - 2) \end{array}\right]$$

$$R_X(\tau) = E\left[X(t)\,X(t+\tau)\right] = \left[\begin{array}{l} 16 \times \frac{1}{2} \times \mathrm{tri}(\tau) + 4 \times \frac{1}{2} \times \mathrm{tri}(\tau) + 16 \times \left[1 - \mathrm{tri}(\tau)\right] \times \frac{1}{4} \\ +8 \times \left[1 - \mathrm{tri}(\tau)\right] \times \frac{1}{4} + 8 \times \left[1 - \mathrm{tri}(\tau)\right] \times \frac{1}{4} + 4 \times \left[1 - \mathrm{tri}(\tau)\right] \times \frac{1}{4} \end{array}\right]$$

$$R_X(\tau) = E\left[X(t)\,X(t+\tau)\right] = 10\,\mathrm{tri}(\tau) + 9\left(1 - \mathrm{tri}(\tau)\right)$$

28. In the system below, the signals, X and N are both bandlimited white noise with a bandwidth of 20 kHz (and mean values of zero). That is,

$$G_X(f) = G_N(f) = 0 \;\;,\;\; |f| > 20\ \text{kHz}.$$

The signals are statistically independent. The variance of N is 10. The variance of the response signal, $Y(t)$, is estimated twice by sampling $Y(t)$ at the Nyquist rate of 40 kHz (making the samples uncorrelated) for 1 second and estimating its variance from those samples. The first time the signal, $X(t)$, is not present and the second time it is present. The results are

Measurement #1 with $X(t)$ not present (set to zero). $\sigma_Y^2 = 200$

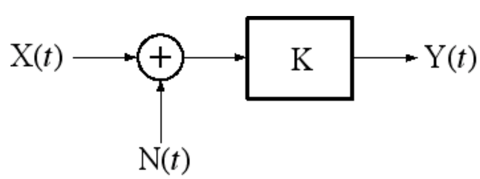
Measurement #2 with $X(t)$ present. $\sigma_Y^2 = 300$

(a) What is your best estimate of the gain constant, K?

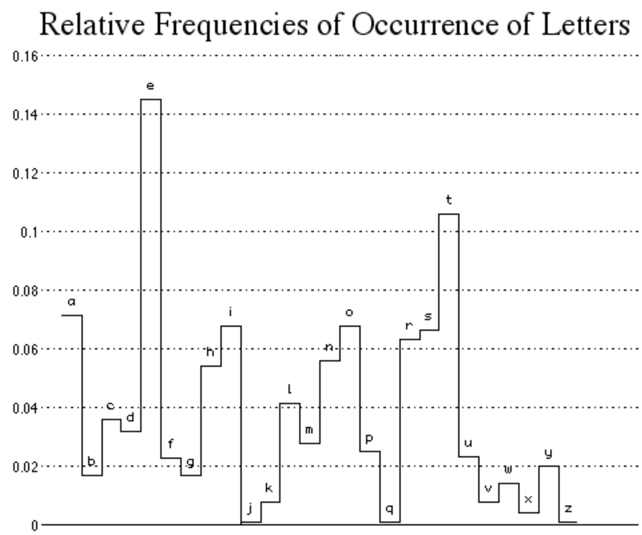
$$\sigma_N^2 K^2 = \sigma_Y^2 \Rightarrow K = \pm\sqrt{\frac{\sigma_Y^2}{\sigma_N^2}} = \sqrt{20} \cong 4.47$$

(b) What is your best estimate of the variance of $X(t)$, σ_X^2?

$$\left(\sigma_N^2 + \sigma_X^2\right)K^2 = \Rightarrow \sigma_X^2 = \frac{\sigma_Y^2}{K^2} - \sigma_N^2 = \frac{300}{20} - 10 = 5$$



29. The frequency of occurrence of the letters of the alphabet in a typical English-language text sample is described by the relative-frequency histogram below.



a , 0.071469	b , 0.017043	c , 0.036016	d , 0.032077	e , 0.14503
f , 0.022751	g , 0.016882	h , 0.054426	i , 0.068012	j ,0.0010451
k , 0.0079588	l , 0.041402	m , 0.027896	n , 0.056275	o , 0.067771
p , 0.025404	q , 0.0010451	r , 0.063269	s , 0.066565	t , 0.10588
u , 0.023394	v , 0.0081196	w , 0.014229	x , 0.004502	y , 0.020339
z , 0.0012059				

Based solely on this relative-frequency histogram,

(a) What is your best estimate of the probability that, in another similar text, 10 randomly selected characters will not contain the letter, “e”?

$$\Pr(\text{single char not being "e"}) = 1 - 0.14503 = 0.85497$$

$$\Pr(\text{10 char's all not being "e"}) = (0.85497)^{10} = 0.2086$$

(b) What is your best estimate of the probability that, in another similar text, 10 randomly selected characters will contain exactlyone letter, “i”?

This is a Bernoulli trial problem.

$$\Pr(1\ \text{"}i\text{" in 10 characters}) = \binom{10}{1}[\Pr(\text{"}i\text{"})]^1[1 - \Pr(\text{"}i\text{"})]^9$$

$$\Pr(1\ \text{"}i\text{" in 10 characters}) = \binom{10}{1}[0.068012]^1[0.931988]^9 = 10(0.068012)(0.5305) = 0.36081$$

30. A coin is flipped until a head appears or N_0 flips have occurred. The number of flips is N. What is $E(N)$?

$$\Pr(N=1)=\frac{1}{2},\ \Pr(N=2)=\frac{1}{4},\ \cdots\Pr(N=N_0-1)=\frac{1}{2^{N_0-1}}$$

$$\Pr(N=N_0)=1-\left[\Pr(N=1)+\Pr(N=2)+\cdots+\Pr(N=N_0-1)\right]=\frac{1}{2^{N_0-1}}$$

$$\mathrm{E}(N)=\sum_{n=1}^{N_0} n\Pr(N=n)=\frac{N_0}{2^{N_0-1}}+\sum_{n=1}^{N_0-1}\frac{n}{2^n}$$

31. A man can go to the office by two routes. Route #1: N_1 minutes to a bridge, N_2 more minutes to the office. Route #2: N_3 minutes to the office. $N_3 > N_1 + N_2$. $\Pr(\text{bridge out}) = p_0$. The man always starts to work by the bridge route. When the bridge is out he returns home and then takes the longer route. What is his expected travel time, $\mathrm{E}(\tau)$?

$$\mathrm{E}(\tau)=(\text{Bridge Route Time})\cdot\Pr(\text{bridge not out})+$$
$$(\text{Twice Time to Bridge}+\text{Non-Bridge Route Time})\cdot\Pr(\text{bridge out})$$

$$\mathrm{E}(\tau)=(N_1+N_2)(1-p_0)+(2N_1+N_3)p_0$$

32. A probability space is $\{A,B,C,D,E,F\}$. All outcomes are equally likely.

(a) Number of events is 2^N where N is the number of outcomes, in this case, 6. Number of events is 64.

(b) Event, X, is $\{A,C,F\}$ and event, Y, is $\{B,C,E\}$. $X \cup Y = \{A,B,C,E,F\}$.

$$\Pr(X \cup Y)=\Pr(A)+\Pr(B)+\Pr(C)+\Pr(E)+\Pr(F)=\frac{5}{6}$$

33. The pdf of X is constant between x_1 and x_2, $x_2 > x_1$, and has an impulse of strength, 0.4, at x_3. Find the expected value of X, E(X).

$$\mathrm{E}(X)=\int_{-\infty}^{\infty} x\,\mathrm{p}_X(x)\,dx = K\int_{x_1}^{x_2} x\,dx+0.4x_3$$

$$K=\frac{0.6}{x_2-x_1}$$

$$\mathrm{E}(X)=\frac{0.6}{x_2-x_1}\int_{x_1}^{x_2} x\,dx+0.4x_3=\frac{0.6}{x_2-x_1}\frac{1}{2}\left(x_2^2-x_1^2\right)+0.4x_3=0.6\frac{x_2+x_1}{2}+0.4x_3$$

34. Two men and two women are selected at random. The probability of a man being left-handed is 0.15. The probability of a woman being left-handed is 0.1. What is the probability that exactly two of these people are left-handed?

$$\Pr(2 \text{ LH people}) = \Pr(0 \text{ LHM})\Pr(2 \text{ LHW}) + \Pr(1 \text{ LHM})\Pr(1 \text{ LHW}) + \Pr(2 \text{ LHM})\Pr(0 \text{ LHW})$$

$$\Pr(0 \text{ LHM})\Pr(2 \text{ LHW}) = (0.85)^2(0.1)^2 = 0.007225$$

$$\Pr(1 \text{ LHM})\Pr(1 \text{ LHW}) = \binom{2}{1}(0.15)^1(0.85)^1\binom{2}{1}(0.1)^1(0.9)^1$$

$$\Pr(1 \text{ LHM})\Pr(1 \text{ LHW}) = 4\left[(0.15)(0.85)(0.1)(0.9)\right] = 0.0459$$

$$\Pr(2 \text{ LHM})\Pr(0 \text{ LHW}) = (0.15)^2(0.9)^2 = 0.018225$$

$$\Pr(2 \text{ LH people}) = 0.007225 + 0.0459 + 0.018225 = 0.07135$$