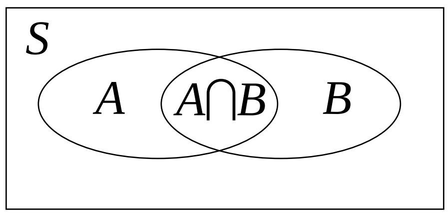
Solution to EE 504 Test #1 F03

1. The city of Trenton has 1000 homes. 600 of the homes subscribe to the Trenton Herald newspaper (Event *A*) and 400 of the homes subscribe to the Trenton Tribune newspaper (Event *B*). 200 homes subscribe to both newspapers (Event $A \cap B$).

(The union of sets 1 and 2 is $S_1 \cup S_2$. The intersection of sets 1 and 2 is $S_1 \cap S_2$. The complement of set, *S* is $\overline{S}$ and the difference between sets 1 and 2 is $S_1 - S_2$.)

A Venn diagram can be very helpful in solving problems if this type.



(a) (10 pts) How many homes do not subscribe to either of these newspapers and how can you describe this event in terms of *A* and *B* and the set operations, union, intersection, complement and difference?

The event is $\overline{A \cup B}$. $\Pr\left(\overline{A \cup B}\right) = 1 - \Pr(A \cup B)$.

$$\Pr(A \cup B) = \underbrace{\Pr(A)}_{0.6} + \underbrace{\Pr(B)}_{0.4} - \underbrace{\Pr(A \cap B)}_{0.2} = 0.8$$

Therefore $\Pr\left(\overline{A \cup B}\right) = 1 - 0.8 = 0.2$ and the number of homes not subscribing to any newspaper is $0.2 \times 1000 = 200$.

This indicates that the total number of homes subscribing to one or more newspapers is 800. It is <u>not</u> 600 + 400 = 1000 because this double counts the homes which subscribe to both newspapers.

(b) (13 pts) How many of the homes subscribe to <u>exactly one</u> newspaper (which could be either one, the Herald or the Tribune, but not both newspapers) and how can you describe this event in terms of *A* and *B* and the set operations, union, intersection, complement and difference?

The event is $\left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right)$.

$$\Pr\left(\left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right)\right) = \Pr\left(A \cap \overline{B}\right) + \Pr\left(B \cap \overline{A}\right) - \underbrace{\Pr\left(\underbrace{\left(A \cap \overline{B}\right) \cap \left(B \cap \overline{A}\right)}_{\varnothing}\right)}_{=0}.$$

$$\Pr\left(A \cap \overline{B}\right) = \underbrace{\Pr(A)}_{0.6} - \underbrace{\Pr(A \cap B)}_{0.2} = 0.4 \qquad \Pr\left(B \cap \overline{A}\right) = \underbrace{\Pr(B)}_{0.4} - \underbrace{\Pr(A \cap B)}_{0.2} = 0.2$$

$$\Pr\left(\left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right)\right) = 0.4 + 0.2 = 0.6$$

Therefore the number of homes subscribing to exactly one newspaper is $0.6 \times 1000 = 600$.

Alternate Solution:

The event is also $(A - B) \cup (B - A)$.

$$\Pr\left((A - B) \cup (B - A)\right) = \Pr(A - B) + \Pr(B - A) - \underbrace{\Pr\left(\underbrace{(A - B) \cap (B - A)}_{=\varnothing}\right)}_{=0}$$

$$\Pr\left((A - B) \cup (B - A)\right) = \underbrace{\Pr\left(A - (A \cap B)\right)}_{0.4} + \underbrace{\Pr\left(B - (A \cap B)\right)}_{0.2} = 0.6$$

Alternate Solution:

The event is also $(A \cup B) - (A \cap B)$. This implies that the number of homes subscribing to exactly one newspaper is the number subscribing to one or more, minus the number subscribing to both. In this case that is 800 - 200 = 600.

(c) (7 pts) If a home is selected at random, what is the numerical probability that it subscribes to both newspapers given that we know it subscribes to at least one?

$$\Pr(A \cap B \mid A \cup B) = \frac{\Pr\left((A \cap B) \cap (A \cup B)\right)}{\Pr(A \cup B)} = \frac{\Pr(A \cap B)}{\underbrace{\Pr(A \cup B)}_{0.8 \text{ from part (a)}}} = \frac{0.2}{0.8} = 0.25$$

This is simply the ratio of the number of homes subscribing to both newspapers to the total number of homes subscribing to at least one newspaper.

2. (6 pts) How many distinguishable permutations of the set of letters, $\{a, b, c, c, d, d, d\}$, can be formed?

This is the number of distinguishable permutations of n things with c classes of indistinguishable members of sizes, $n_1, n_2, \cdots n_c$, which is

$$\frac{n!}{n_1! n_2! \cdots n_c!} = \frac{7!}{1! \times 1! \times 2! \times 3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 1 \times 2 \times 1 \times 3 \times 2 \times 1} = 7 \times 6 \times 5 \times 2 = 420$$

7! is the number of permutations of 7 objects in 7 positions. Since c and c are indistinguishable and there are 2! ways of permuting them in two positions and d, d and d are also indistinguishable and there are 3! ways of permuting them in three positions, the number of <u>distinguishable</u> permutations of these 7 objects is the number of permutations of

7 objects divided by the product of the number of permutations of 2 objects and the number of permutations of 3 objects.

3. (16 pts) An experiment consists of tossing a fair die (six sides and the numbers 1 through 6 on its faces) until the number, 3, occurs or the die is tossed 4 times, whichever comes first. The number of tosses is a random variable, N. What is the numerical expected value of N, $\mathrm{E}(N)$?

All the unique outcomes are:

1. 3 occurs on the first toss - Probability = $\frac{1}{6}$.

2. 3 occurs first on the second toss - Probability = $\frac{5}{6}\times\frac{1}{6}=\frac{5}{36}$.

3. 3 occurs first on the third toss - Probability = $\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{216}$

4. A fourth toss is required - Probability = $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{216}$

$$\mathrm{E}(N)=\sum_{i=1}^{4} N_i \Pr(N=N_i)=1\times\frac{1}{6}+2\times\frac{5}{36}+3\times\frac{25}{216}+4\times\frac{125}{216}=\frac{36+60+75+500}{216}=3.1065$$

Two answers appearing on test papers that were considered absurd were an N less than 1 and an N greater than 30.

One person found the expected value of the sum of the numbers tossed instead of the expected value of the number of tosses.

One person reasoned that if the probability of a 3 on one toss is 1/6 that the probability of a 3 in 4 tosses is 4/6. Following that logic, the probability of a 3 in 7 tosses would be 7/6, a number greater than one, therefore absurd. The probability of exactly one 3 in 4 tosses is

$$\Pr(\text{Eactly one 3 in 4 tosses})=\binom{4}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3=4\times\frac{1}{6}\times\frac{125}{216}=0.3858\ .$$

The probability of exactly one 3 in 7 tosses is

$$\Pr(\text{Eactly one 3 in 7 tosses})=\binom{7}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^6=0.3907$$

4. A random variable, X, has a uniform probability density function (pdf) between, -3 and +6. Another random variable, Y, is created through the function graphed below relating Y to X.

(a) (3 pts) What is the numerical probability of the event, $X=2$?

Zero. The pdf of X is continuous.

Any answer other than zero indicates a lack of understanding of the meaning of a continuous pdf.

(b) (4 pts) What is the numerical probability of the event, $Y = 2$?

The event, $Y = 2$, is caused by the event, $\{2 < X \le 6\}$. Therefore

$$\Pr(Y = 2) = \Pr(\{2 < X \le 6\}) = \int_2^6 \mathrm{p}_X(x)dx = \frac{1}{9}(6-2) = \frac{4}{9} = 0.4444\ldots$$

Many students answered zero here also. But the pdf of Y contains impulses and therefore the probability of an exact value is not necessarily zero.

(c) (10 pts) Sketch the pdf of Y. (I recommend you <u>not</u> use the formula,

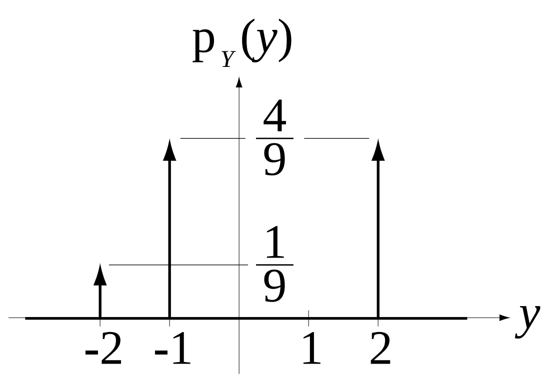
$$\mathrm{p}_Y(y) = \frac{\mathrm{p}_X(x_1)}{\left|\left(\frac{dY}{dX}\right)_{X=x_1}\right|} + \frac{\mathrm{p}_X(x_2)}{\left|\left(\frac{dY}{dX}\right)_{X=x_2}\right|} + \cdots + \frac{\mathrm{p}_X(x_N)}{\left|\left(\frac{dY}{dX}\right)_{X=x_N}\right|} \ ,$$

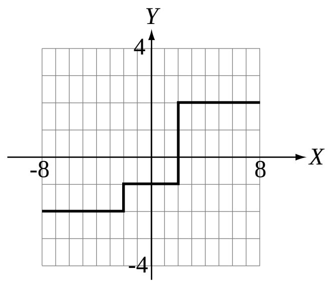
but simply reason from the probabilities of corresponding events in X and Y.)

Y can only take one exactly three values, -1, -1 and +2.

$$\mathrm{p}_Y(y) = \underbrace{\Pr(Y=-2)}_{\Pr(-3<X\le-2)=\frac{1}{9}}\delta(y+2) + \underbrace{\Pr(Y=-1)}_{\Pr(-2<X\le2)=\frac{4}{9}}\delta(y+1) + \underbrace{\Pr(Y=2)}_{\Pr(2<X\le6)=\frac{4}{9}}\delta(y-2)$$

$$\mathrm{p}_Y(y) = \frac{1}{9}\delta(y+2) + \frac{4}{9}\delta(y+1) + \frac{4}{9}\delta(y-2)$$





5. A random variable, X, has a pdf, $\mathrm{p}_X(x) = 0.2\delta(x) + 0.5\delta(x-4) + 0.3\delta(x-6)$.

(a) (6 pts) What is its numerical expected value, $\mathrm{E}(X)$?

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{p}_X(x)dx = \int_{-\infty}^{\infty} x[0.2\delta(x) + 0.5\delta(x-4) + 0.3\delta(x-6)]dx$$

$$= 0.2\times 0 + 0.5\times 4 + 0.3\times 6 = 3.8$$

(b) (6 pts) What is its numerical variance, σ_X^2 ?

$$\sigma_X^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2\,\mathrm{p}_X(x)dx = \int_{-\infty}^{\infty} x^2[0.2\delta(x) + 0.5\delta(x-4) + 0.3\delta(x-6)]dx$$

$$= 0.2\times 0^2 + 0.5\times 4^2 + 0.3\times 6^2 = 18.8$$

$$\sigma_X^2 = 18.8 - 3.8^2 = 4.36$$