Solution to EE 504 Test #2 F03

1. Three random variables, X, Y and Z have the following characteristics:

$$\mathrm{E}(X) = 1 \ , \ \mathrm{E}(Y) = 0 \ , \ \mathrm{E}(Z) = -3$$
$$\sigma_X^2 = 4 \ , \ \sigma_Y^2 = 4 \ , \ \sigma_Z^2 = 1$$
$$\sigma_{XY} = 0 \ , \ \sigma_{XZ} = 2 \ , \ \sigma_{YZ} = 1$$

Find the numerical value of the correlation coefficient, ρ_{XW}, between X and $W = X - 2Y + Z$.

$$\rho_{XW} = \frac{\sigma_{XW}}{\sigma_X \sigma_W}$$
$$\sigma_X = \sqrt{\sigma_X^2} = 2$$

Using the formula for the variance of a linear combination, Z, of N random variables, X_i,

$$\sigma_Z^2 = \sum_{i=1}^{N} a_i^2 \sigma_{X_i}^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} a_i a_j \sigma_{X_i X_j} \ ,$$

we get

$$\sigma_W^2 = (1)^2 \sigma_X^2 + (-2)^2 \sigma_Y^2 + (1)^2 \sigma_Z^2 + 2(1)(-2)\sigma_{XY} + 2(1)(1)\sigma_{XZ} + 2(-2)(1)\sigma_{YZ}$$
$$\sigma_W^2 = 4 + 4 \times 4 + 1 - 4 \times 0 + 2 \times 2 - 4 \times 1 = 21 \Rightarrow \sigma_W = \sqrt{21}$$

The variance of W could also be found using

$$\sigma_W^2 = \mathrm{E}(W^2) - [\mathrm{E}(W)]^2$$
$$\mathrm{E}(W) = \mathrm{E}(X - 2Y + Z) = \mathrm{E}(X) - 2\,\mathrm{E}(Y) + \mathrm{E}(Z) = 1 - 2(0) - 3 = -2$$
$$\mathrm{E}(W^2) = \mathrm{E}\left((X - 2Y + Z)^2\right) = \mathrm{E}(X^2) + 4\,\mathrm{E}(Y^2) + \mathrm{E}(Z^2) - 4\,\mathrm{E}(XY) + 2\,\mathrm{E}(XZ) - 4\,\mathrm{E}(YZ)$$
$$\mathrm{E}(X^2) = \sigma_X^2 + [\mathrm{E}(X)]^2 = 4 + 1^2 = 5$$
$$\mathrm{E}(Y^2) = \sigma_Y^2 + [\mathrm{E}(Y)]^2 = 4 + 0^2 = 4$$
$$\mathrm{E}(Z^2) = \sigma_Z^2 + [\mathrm{E}(Z)]^2 = 1 + (-3)^2 = 10$$
$$\mathrm{E}(XY) = \sigma_{XY} + \mathrm{E}(X)\mathrm{E}(Y) = 0 + 1(0) = 0$$
$$\mathrm{E}(XZ) = \sigma_{XZ} + \mathrm{E}(X)\mathrm{E}(Z) = 2 + 1(-3) = -1$$
$$\mathrm{E}(YZ) = \sigma_{YZ} + \mathrm{E}(Y)\mathrm{E}(Z) = 1 + 0(-3) = 1$$
$$\mathrm{E}(W^2) = 5 + 4(4) + 10 - 4(1)(0) + 2(1)(-3) - 4(0)(-3) = 5 + 16 + 10 - 6 = 25$$
$$\sigma_W^2 = 25 - (-2)^2 = 21 \Rightarrow \sigma_W = \sqrt{21} \quad \text{Check.}$$

The covariance of X with W is

$$\sigma_{XW} = \mathrm{E}(XW) - \mathrm{E}(X)\mathrm{E}(W)$$
$$\sigma_{XW} = \mathrm{E}\left(X(X - 2Y + Z)\right) - \mathrm{E}(X)\mathrm{E}(X - 2Y + Z)$$

$$\sigma_{XW} = \mathrm{E}\left(X^2 - 2XY + XZ\right) - 1\times(1-0-3)$$

$$\sigma_{XW} = \underbrace{\mathrm{E}\left(X^2\right)}_{5} - 2\underbrace{\mathrm{E}(XY)}_{0} + \underbrace{\mathrm{E}(XZ)}_{-1} + 2$$

$$\sigma_{XW} = 5 - 1 + 2 = 6$$

$$\rho_{XW} = \frac{6}{2\times\sqrt{21}} = 0.6547$$

2. Two independent random variables, X and Y, have pdf's given by

$$\mathrm{p}_X(x) = 0.3\delta(x-2) + 0.7\delta(x+1)$$

and $\mathrm{p}_Y(y)$ is uniform over the range, $2 < y < 7$.

(a) Sketch the pdf of $Z = X - Y$.

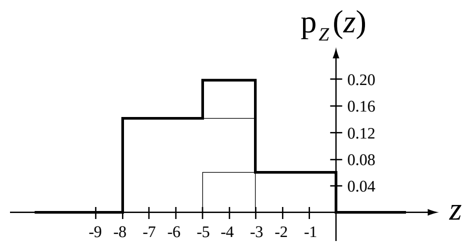
$$\mathrm{p}_Y(y) = \frac{1}{5}\mathrm{rect}\left(\frac{y-4.5}{5}\right)$$

$$\mathrm{p}_{(-Y)}(y) = \mathrm{p}_Y(-y) = \frac{1}{5}\mathrm{rect}\left(\frac{-y-4.5}{5}\right) = \frac{1}{5}\mathrm{rect}\left(\frac{y+4.5}{5}\right)$$

$$\mathrm{p}_Z(\mathrm{z}) = \mathrm{p}_X(\mathrm{z}) * \mathrm{p}_{(-Y)}(z) = \left[0.3\delta(\mathrm{z}-2) + 0.7\delta(\mathrm{z}+1)\right] * \frac{1}{5}\mathrm{rect}\left(\frac{z+4.5}{5}\right)$$

$$\mathrm{p}_Z(z) = \frac{1}{5}\left[0.3\delta(\mathrm{z}-2) * \mathrm{rect}\left(\frac{z+4.5}{5}\right) + 0.7\delta(z+1) * \mathrm{rect}\left(\frac{z+4.5}{5}\right)\right]$$

$$\mathrm{p}_Z(z) = \frac{1}{5}\left[0.3\mathrm{rect}\left(\frac{z+2.5}{5}\right) + 0.7\mathrm{rect}\left(\frac{z+5.5}{5}\right)\right]$$



(b) What is the numerical probability that $|Z| < 4$?

$$\Pr\left(|Z| < 4\right) = \Pr(-4 < Z < 4) = \int_{-4}^{4} \mathrm{p}_Z(z)\,dz$$

$$\Pr\left(|Z| < 4\right) = \frac{1}{5}\int_{-4}^{4}\left[0.3\mathrm{rect}\left(\frac{z+2.5}{5}\right) + 0.7\mathrm{rect}\left(\frac{z+5.5}{5}\right)\right]dz$$

$$\Pr(|Z| < 4) = \frac{1}{5}\int_{-4}^{4} 0.3\,\mathrm{rect}\left(\frac{z+2.5}{5}\right)dz + \frac{1}{5}\int_{-4}^{4} 0.7\,\mathrm{rect}\left(\frac{z+5.5}{5}\right)dz$$

$$\Pr(|Z| < 4) = \frac{0.3}{5}\int_{-4}^{0} dz + \frac{0.7}{5}\int_{-4}^{-3} dz = \frac{0.3}{5}(4) + \frac{0.7}{5}(1) = \frac{1.2+0.7}{5} = 0.38$$

Referring to the pdf graph, this is the area under the pdf from -4 to 0, which is 0.2(1) + 0.06(3) = 0.38. Check.

3. An experiment is performed 200 times. The sample mean, $\bar{x}$, of the experimental outcomes is 25 and the sample standard deviation, S_X, is 4. A sample size of 200 is large enough to reasonably use the sample standard deviation as the population standard deviation, σ_X, and to reasonably assume that the pdf of the sample mean, $\bar{X}$, is Gaussian. A confidence interval on the sample mean is to be reported in the form, $25 \pm k$, for a confidence level of 80%. Find the numerical value of *k*.

The variance of the mean is $\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{N} = \frac{16}{200} = 0.08 \Rightarrow \sigma_{\bar{X}} = \sqrt{0.08} = 0.2828$.

From the normal distribution tables, for a confidence level of 80%, the confidence interval should be approximately $25 \pm 1.3\sigma_{\bar{X}} = 25 \pm 0.3676$. Therefore *k* is 0.3676.

4. What characteristic of its probability density function indicates that a random variable is continuous?

The pdf has <u>no</u> impulses in it.

OR

What characteristic of its probability density function indicates that a random variable is discrete?

The pdf has <u>only</u> impulses in it.

5. A deterministic random process has sample functions of the form,

$$\mathrm{X}(t) = A\cos(2\pi t + \theta),$$

where *A* and θ are random over the ensemble but constant for any single sample function. Let θ be uniformly distributed over the range, $-\pi < \theta < \pi$ and let *A* be Gaussian distributed with an expected value of 1 and a variance of 4. Let *A* and θ be independent, implying that *A* and $\cos(2\pi t + \theta)$ are also independent. Find the mean-squared value of the random process, $\mathrm{E}\left(X^2(t)\right)$.

$$\mathrm{E}\left(X^2\right) = \mathrm{E}\left(A^2\cos^2(2\pi t + \theta)\right) = \mathrm{E}\left(\frac{A^2}{2}\left(1 + \cos(4\pi t + 2\theta)\right)\right)$$

$$\mathrm{E}\left(X^2\right)=\mathrm{E}\left(\frac{A^2}{2}\right)+\mathrm{E}\left(\frac{A^2}{2}\cos(4\pi t+2\theta)\right)$$

Since A and $\cos(2\pi t+\theta)$ (and, by implication, $\cos(4\pi t+2\theta)$) are independent,

$$\mathrm{E}\left(X^2\right)=\mathrm{E}\left(\frac{A^2}{2}\right)+\mathrm{E}\left(\frac{A^2}{2}\right)\mathrm{E}\left(\cos(4\pi t+2\theta)\right)$$

$$\mathrm{E}\left(\cos(4\pi t+2\theta)\right)=\int_{-\infty}^{\infty}\cos(4\pi t+2\theta)\,p_\Theta(\theta)\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(4\pi t+2\theta)\,d\theta$$

$$\mathrm{E}\left(\cos(4\pi t+2\theta)\right)=\frac{1}{2\pi}\left[\frac{\sin(4\pi t+2\theta)}{2}\right]_{-\pi}^{\pi}=\frac{1}{2\pi}\left[\frac{\sin(4\pi t+2\pi)}{2}-\frac{\sin(4\pi t-2\pi)}{2}\right]$$

$$\mathrm{E}\left(\cos(4\pi t+2\theta)\right)=\frac{1}{4\pi}\left[\underbrace{\sin(4\pi t+2\pi)}_{=\sin(4\pi t)}-\underbrace{\sin(4\pi t-2\pi)}_{=\sin(4\pi t)}\right]=0$$

(This result, $\mathrm{E}\left(\cos(4\pi t+2\theta)\right)=0$, should be obvious but the preceding three lines prove it in case it is not.) Then

$$\mathrm{E}\left(X^2\right)=\mathrm{E}\left(\frac{A^2}{2}\right)=\frac{1}{2}\mathrm{E}\left(A^2\right)=\frac{1}{2}\left(\sigma_A^2+\left[\mathrm{E}(A)\right]^2\right)=\frac{1}{2}(4+1)=\frac{5}{2}=2.5$$