

Solution to EE 504 Test #3 F03

1. Below are some functions which are candidates for being valid autocorrelation functions. For each one indicate by circling the correct answer whether it is possible or impossible that the function could be the autocorrelation function of a real random variable and, if it is impossible, state why in the space provided.

(a) $R_x(\tau) = 10\text{sinc}(20\tau)$ Possible

Reason: Even function, Maximum at zero, Fourier transform is non-negative

(b) $R_x(\tau) = 100(u(\tau - 4) - u(\tau + 4))$ Impossible

Reason: Value at $\tau = 0$ is negative. Fourier transform is not non-negative everywhere.

(c) $R_x(\tau) = 100(u(\tau - 2) - u(\tau + 6))$ Impossible

Reason: Not an even function.

(d) $R_x(\tau) = 20\cos(400\pi\tau)$ Possible

Reason: Even function, Maximum at zero, Fourier transform is non-negative

(e) $R_x(\tau) = 5e^{-3\tau} \sin(25\pi\tau)$ Impossible

Reason: Not an even function

(f) $R_x(\tau) = 10\text{tri}(\tau) - 6\text{tri}(2\tau)$ Impossible

Reason: Maximum not at zero.

(g) $R_x(\tau) = 3\text{tri}(\tau) + 4\cos(10\pi\tau) - 2$ Impossible

Reason: Fourier transform is not non-negative everywhere. Also, using the rule for finding the expected value of X would produce an imaginary number.

2. The autocorrelation of a random process is $R_x(\tau) = 3\text{tri}(\tau) + 4\cos(10\pi\tau) + 2$.

(a) What is the expected value, $E(X)$?

$$E(X) = \lim_{\tau \rightarrow \infty} (\pm \sqrt{3\text{tri}(\tau) + 2}) = \pm\sqrt{2} = \pm 1.414$$

(b) What is the mean-squared value, $E(X^2)$?

$$E(X^2) = R_x(0) = 3 + 4 + 2 = 9$$

3. A finite-time integrator has an impulse response,

$$h(t) = -10[u(t) - u(t-4)] = -10\text{rect}\left(\frac{t-2}{4}\right).$$

Noise, X , with a double-sided PSD of $G_X(f) = 20 + 4\delta(f)$ and a positive expected value excites (is the input signal to) this finite-time integrator.

(a) The expected value, $E(Y)$, of the response (output signal), Y .

$$R_X(\tau) = 20\delta(\tau) + 4$$

$$E(Y) = E(X) \int_{-\infty}^{\infty} h(t) dt$$

$$E(X) = \lim_{\tau \rightarrow \infty} \left(\sqrt{\overline{R_X(\tau)}} \right) = 2$$

$$\int_{-\infty}^{\infty} h(t) dt = -40$$

Therefore

$$E(Y) = 2(-40) = -80$$

Alternate Solution:

$$E(Y) = \lim_{\tau \rightarrow \infty} \left(\pm \sqrt{\overline{R_Y(\tau)}} \right)$$

$$R_Y(\tau) = F^{-1}(G_Y(f))$$

$$G_Y(f) = G_X(f) |H(f)|^2$$

$$H(f) = -40 \text{sinc}(4f) e^{-j4\pi f} \Rightarrow |H(f)|^2 = 1600 \text{sinc}^2(4f)$$

$$G_Y(f) = [20 + 4\delta(f)] 1600 \text{sinc}^2(4f) = [32000 \text{sinc}^2(4f) + 6400\delta(f)]$$

$$R_Y(\tau) = \frac{32000}{4} \text{tri}\left(\frac{t}{4}\right) + 6400$$

$$E(Y) = \lim_{\tau \rightarrow \infty} \left(\pm \sqrt{\frac{32000}{4} \text{tri}\left(\frac{t}{4}\right) + 6400} \right) = \pm 80$$

This gets the expected value of Y to within a sign but does not resolve the sign like the first method does.

(b) The mean-squared value, $E(Y^2)$, of the response, Y .

$$\begin{aligned}
E(Y^2) &= R_Y(0) \\
R_Y(\tau) &= R_X(\tau) * h(\tau) * h(-\tau) \\
h(\tau) * h(-\tau) &= \left[-10 \left(\text{rect} \left(\frac{\tau-2}{4} \right) \right) \right] * \left[-10 \left(\text{rect} \left(\frac{-\tau-2}{4} \right) \right) \right] \\
h(\tau) * h(-\tau) &= 100 \text{rect} \left(\frac{\tau-2}{4} \right) * \text{rect} \left(\frac{\tau+2}{4} \right) \\
h(\tau) * h(-\tau) &= 100 \left[4 \text{tri} \left(\frac{\tau}{4} \right) \right] \\
R_Y(\tau) &= [20\delta(\tau) + 4] * 400 \text{tri} \left(\frac{\tau}{4} \right) \\
R_Y(\tau) &= 400 \left[20\delta(\tau) * \text{tri} \left(\frac{\tau}{4} \right) + 4 * \text{tri} \left(\frac{\tau}{4} \right) \right] \\
R_Y(\tau) &= 400 \left[20 \text{tri} \left(\frac{\tau}{4} \right) + 16 \right] \\
R_Y(0) &= 400 [20 \text{tri}(0) + 16] = 400 \times 36 = 14400
\end{aligned}$$

Alternate Solution:

$$\begin{aligned}
E(Y^2) &= \int_{-\infty}^{\infty} G_Y(f) df = \int_{-\infty}^{\infty} G_X(f) |H(f)|^2 df \\
H(f) &= -40 \text{sinc}(4f) e^{-j4\pi f} \Rightarrow |H(f)|^2 = 1600 \text{sinc}^2(4f) \\
E(Y^2) &= \int_{-\infty}^{\infty} [20 + 4\delta(f)] 1600 \text{sinc}^2(4f) df = 1600 \int_{-\infty}^{\infty} [20 + 4\delta(f)] \text{sinc}^2(4f) df \\
E(Y^2) &= 1600 \left[\int_{-\infty}^{\infty} 20 \text{sinc}^2(4f) df + \int_{-\infty}^{\infty} 4\delta(f) \text{sinc}^2(4f) df \right] \\
E(Y^2) &= 32000 \underbrace{\int_{-\infty}^{\infty} \text{sinc}^2(4f) df}_{= \frac{1}{4} \text{tri} \left(\frac{t}{4} \right)_{t=0} = \frac{1}{4}} + 6400 \int_{-\infty}^{\infty} \delta(f) \text{sinc}^2(4f) df = 8000 + 6400 = 14400
\end{aligned}$$

4. A circuit consists of a resistor, R , at an absolute temperature, T , and a capacitor, C , in parallel. (The double-sided PSD of the short-circuit Johnson noise current of a resistor is $\frac{2kT}{R}$ and the double-sided PSD of the open-circuit Johnson noise voltage is $2kTR$ where k is Boltzmann's constant, $1.38 \times 10^{-23} \frac{\text{J}}{\text{K}}$.)

(a) Find a general formula for the mean-squared voltage across the capacitor (and the resistor) in terms of k , R , T and C . This formula should contain no integral signs, Fourier transforms or convolution operators. You may use the integration formula,

$$\int \frac{dx}{a^2 + (bx)^2} = \frac{1}{ab} \tan^{-1}\left(\frac{bx}{a}\right), \text{ if needed during the process of finding the formula.}$$

Using a current source model for the Johnson noise of the resistor, the transfer function from the noise current to the capacitor voltage is the impedance of the parallel R-C circuit. The mean-squared noise voltage is

$$\begin{aligned} E(V^2) &= \int_{-\infty}^{\infty} G_{ov}(f) df \\ G_{ov}(f) &= G_i(f) |H_i(f)|^2 = G_i(f) H_i^*(f) H_i(f) \\ G_i(f) &= \frac{2kT}{R} \\ H_i(f) &= \frac{\frac{R}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} = \frac{R}{j2\pi fRC + 1} \\ G_{ov}(f) &= \frac{2kT}{R} \left(\frac{R}{j2\pi fRC + 1} \right) \left(\frac{R}{-j2\pi fRC + 1} \right) \\ G_{ov}(f) &= \frac{2kT}{R} \frac{R^2}{(2\pi fRC)^2 + 1} = \frac{2kTR}{(2\pi RC)^2} \frac{1}{f^2 + \left(\frac{1}{2\pi RC}\right)^2} \\ E(V^2) &= \frac{2kTR}{(2\pi RC)^2} \int_{-\infty}^{\infty} \frac{df}{f^2 + \left(\frac{1}{2\pi RC}\right)^2} \\ E(V^2) &= \frac{kT}{2\pi^2 RC^2} [2\pi RC \tan^{-1}(2\pi RCf)]_{-\infty}^{\infty} = \frac{2\pi RCkT}{2\pi^2 RC^2} \pi = \frac{kT}{C} \end{aligned}$$

Alternate Solution:

If we use a voltage source model for the resistor noise,

$$G_{ov}(f) = G_v(f) |H_v(f)|^2 = G_v(f) H_v^*(f) H_v(f)$$

$$G_v(f) = 2kTR$$

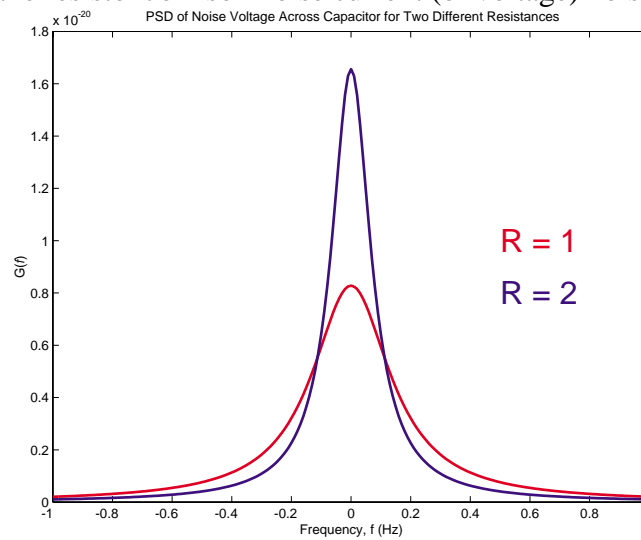
$$H_v(f) = \frac{1}{R + \frac{1}{j2\pi fC}} = \frac{1}{j2\pi fRC + 1}$$

$$G_{ov}(f) = 2kTR \left(\frac{1}{j2\pi fRC + 1} \right) \left(\frac{1}{-j2\pi fRC + 1} \right)$$

$$G_{ov}(f) = 2kTR \frac{1}{(2\pi fRC)^2 + 1} = \frac{2kTR}{(2\pi RC)^2} \frac{1}{f^2 + \left(\frac{1}{2\pi RC}\right)^2}$$

From this step on, the solution is the same as the previous one.

(b) You should find that the mean-squared voltage is independent of resistance, R . By sketching the PSD of the noise voltage across the capacitor for the two cases, $R=1, C=1, T=300$ and $R=2, C=1, T=300$, explain how that can happen even though the PSD of the resistor Johnson noise current (or voltage) noise is a function of R .



The two PSD's have different peak amplitudes and different shapes but their areas are the same.