Solution to ECE 504 Test #2 F02

1. A random variable, X, has a pdf that is uniform between x_1 and x_2. What is the numerical probability that $X < x_0$, given that $X > x_3$?

$$\Pr(X < x_0 \mid X > x_3) = \frac{\Pr(x_3 < X < x_0)}{\Pr(X > x_3)} = \frac{F_X(x_0) - F_X(x_3)}{1 - F_X(x_3)} = \frac{\int_{x_3}^{x_0} p_X(x)dx}{\int_{x_3}^{\infty} p_X(x)dx}$$

$$\Pr(X < x_0 \mid X > x_3) = \frac{\frac{1}{x_2 - x_1}(x_0 - x_3)}{\frac{1}{x_2 - x_1}(x_2 - x_3)} = \frac{x_0 - x_3}{x_2 - x_3}$$

2. A random variable, X, has an expected value of $E(X)$ and a variance, σ_X^2. If a number is computed by averaging N randomly-chosen values of X, what is the probability that the number will be between $E(X) - x_0$ and $E(X) + x_0$?

$$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

If N is large, the central limit theorem applies and the pdf of $\overline{X}$ is Guassian with $E(\overline{X}) = E(X)$ and $\sigma_{\overline{X}}^2 = \frac{\sigma_X^2}{N}$. The probability is

$$\Pr(E(X) - x_0 < \overline{X} < E(X) + x_0) = \int_{E(X)-x_0}^{E(X)+x_0} \frac{1}{\sqrt{2\pi}\sigma_{\overline{X}}} e^{-\frac{(x - E(X))^2}{2\sigma_{\overline{X}}^2}} dx = \frac{2}{\sqrt{2\pi}\sigma_{\overline{X}}} \int_{E(X)}^{E(X)+x_0} e^{-\frac{(x - E(X))^2}{2\sigma_{\overline{X}}^2}} dx$$

Let $u = \frac{x - E(X)}{\sqrt{2}\sigma_{\overline{X}}} \Rightarrow du = \frac{dx}{\sqrt{2}\sigma_{\overline{X}}}$.

Then

$$\Pr(E(X) - x_0 < \overline{X} < E(X) + x_0) = \frac{2\sqrt{2}\sigma_{\overline{X}}}{\sqrt{2\pi}\sigma_{\overline{X}}} \int_0^{\frac{x_0}{\sqrt{2}\sigma_{\overline{X}}}} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x_0}{\sqrt{2}\sigma_{\overline{X}}}} e^{-u^2} du \quad .$$

$$\Pr\left(\mathrm{E}(X) - x_0 < \overline{X} < \mathrm{E}(X) + x_0\right) = \mathrm{erf}\left(\frac{x_0}{\sqrt{2}\sigma_{\overline{X}}}\right)$$

3. The pdf of X is of the form, $\mathrm{p}_X(x) = \dfrac{\mathrm{rect}\left(\dfrac{x - \mathrm{E}(X)}{x_2 - x_1}\right)}{x_2 - x_1}$ and the pdf of Y is of the form, $\mathrm{p}_Y(y) = a\delta(y - y_1) + b\delta(y - y_2)$, $a + b = 1$. The random variable, Z, is $Z = X - Y$. So the pdf of Z is the convolution of the pdf's of X and -Y. The pdf of -Y is $a\delta(-y - y_1) + b\delta(-y - y_2) = a\delta(y + y_1) + b\delta(y + y_2)$. Therefore the pdf of Z is

$$\mathrm{p}_Z(z) = \frac{\mathrm{rect}\left(\dfrac{z - \mathrm{E}(X)}{x_2 - x_1}\right)}{x_2 - x_1} * \left[a\delta(z + y_1) + b\delta(z + y_2)\right]$$

$$\mathrm{p}_Z(z) = a\frac{\mathrm{rect}\left(\dfrac{z + y_1 - \mathrm{E}(X)}{x_2 - x_1}\right)}{x_2 - x_1} + b\frac{\mathrm{rect}\left(\dfrac{z + y_2 - \mathrm{E}(X)}{x_2 - x_1}\right)}{x_2 - x_1} \quad .$$

The probability is the area under this function for $z > z_0$. This is the sum of two areas, one for each of the two terms above. The area under the first term is

$$\frac{a}{x_2 - x_1}(z_2 - z_1)$$

where z_2 is the greater of $\mathrm{E}(X) - y_1 + \dfrac{x_2 - x_1}{2}$ and z_0 and z_1 is the greater of $\mathrm{E}(X) - y_1 - \dfrac{x_2 - x_1}{2}$ and z_0. this function for $z > z_0$. This is the sum of two areas, one for each of the two terms above. The area under the second term is

$$\frac{b}{x_2 - x_1}(z_4 - z_3)$$

where z_4 is the greater of $\mathrm{E}(X) - y_2 + \dfrac{x_2 - x_1}{2}$ and z_0 and z_3 is the greater of $\mathrm{E}(X) - y_2 - \dfrac{x_2 - x_1}{2}$ and z_0. (This last result is best seen by drawing a sketch of the pdf of Z.)

4. Two random variables, X and Y, have a joint pdf, which is one-half in the regions,

$$0 < x < 1 \ , \ 0 < y < 1 \qquad \text{and} \qquad -1 < x < 0 \ , \ -1 < y < 0$$

and zero elsewhere. That is, the joint pdf is

$$\mathrm{p}_{XY}(x,y) = \frac{1}{2}\left[\mathrm{rect}\left(x-\frac{1}{2}\right)\mathrm{rect}\left(y-\frac{1}{2}\right)+\mathrm{rect}\left(x+\frac{1}{2}\right)\mathrm{rect}\left(y+\frac{1}{2}\right)\right] \ .$$

What is the numerical value of the correlation coefficient, ρ_{XY} ?

$$\rho_{XY} = \frac{\mathrm{E}(XY)-\mathrm{E}(X)\mathrm{E}(Y)}{\sigma_X\sigma_Y}$$

$$\mathrm{E}(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{2}\left[\mathrm{rect}\left(x-\frac{1}{2}\right)\mathrm{rect}\left(y-\frac{1}{2}\right)+\mathrm{rect}\left(x+\frac{1}{2}\right)\mathrm{rect}\left(y+\frac{1}{2}\right)\right]dxdy$$

$$\mathrm{E}(X) = \int_0^1\int_0^1\frac{x}{2}dxdy + \int_{-1}^0\int_{-1}^0\frac{x}{2}dxdy = \frac{1}{4}-\frac{1}{4} = 0$$

Similarly,

$$\mathrm{E}(Y) = 0$$

$$\mathrm{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{xy}{2}\left[\mathrm{rect}\left(x-\frac{1}{2}\right)\mathrm{rect}\left(y-\frac{1}{2}\right)+\mathrm{rect}\left(x+\frac{1}{2}\right)\mathrm{rect}\left(y+\frac{1}{2}\right)\right]dxdy$$

$$\mathrm{E}(XY) = \int_0^1\int_0^1\frac{xy}{2}dxdy + \int_{-1}^0\int_{-1}^0\frac{xy}{2}dxdy = \frac{1}{2}\left[\int_0^1 y\int_0^1 xdxdy + \int_{-1}^0 y\int_{-1}^0 xdxdy\right] = \frac{1}{2}\left[\frac{1}{4}+\frac{1}{4}\right] = \frac{1}{4}$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{2}\left[\mathrm{rect}\left(x-\frac{1}{2}\right)\mathrm{rect}\left(y-\frac{1}{2}\right)+\mathrm{rect}\left(x+\frac{1}{2}\right)\mathrm{rect}\left(y+\frac{1}{2}\right)\right]dxdy$$

$$\mathrm{E}(X^2) = \int_0^1\int_0^1\frac{x^2}{2}dxdy + \int_{-1}^0\int_{-1}^0\frac{x^2}{2}dxdy = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

Similarly,

$$\mathrm{E}(Y^2) = \frac{1}{3} \ .$$

Therefore $\sigma_X^2 = \frac{1}{3}$ and $\sigma_Y^2 = \frac{1}{3}$ and $\sigma_X \sigma_Y = \frac{1}{3}$.

Finally, $\rho_{XY} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$

5. A random variable, X, has an expected value, $\mathrm{E}(X)$, and a standard deviation, σ_X. Another random variable, Y, has an expected value, $\mathrm{E}(Y)$, and a standard deviation, σ_Y. X and Y are independent. Two other random variables are formed by $Z_1 = X + Y$ and $Z_2 = X - Y$. Find the numerical value of the covariance, σ_{12}, between Z_1 and Z_2.

$$\sigma_{12} = \mathrm{E}(Z_1 Z_2) - \mathrm{E}(Z_1)\mathrm{E}(Z_2)\ .$$

$$\mathrm{E}(Z_1) = \mathrm{E}(X) + \mathrm{E}(Y) \quad \text{and} \qquad \mathrm{E}(Z_2) = \mathrm{E}(X) - \mathrm{E}(Y)$$

$$\mathrm{E}(Z_1 Z_2) = \mathrm{E}((X+Y)(X-Y)) = \mathrm{E}(X^2 - Y^2) = \mathrm{E}(X^2) - \mathrm{E}(Y^2)$$

$$\sigma_{12} = \mathrm{E}(X^2) - \mathrm{E}(Y^2) - [\mathrm{E}(X) + \mathrm{E}(Y)][\mathrm{E}(X) - \mathrm{E}(Y)]$$

$$\sigma_{12} = \mathrm{E}(X^2) - \mathrm{E}(Y^2) - [\mathrm{E}(X)]^2 + [\mathrm{E}(Y)]^2 = \sigma_X^2 - \sigma_Y^2$$