Web Appendix C - Example of Linearizing and Small Signal Analysis

Even if a system is statically non-linear, linear system analysis techniques may still be useful in analyzing it. As a simple example of using linear system analysis on non-linear systems consider a circuit containing a voltage source, a resistor and a diode in series (Figure C-1).
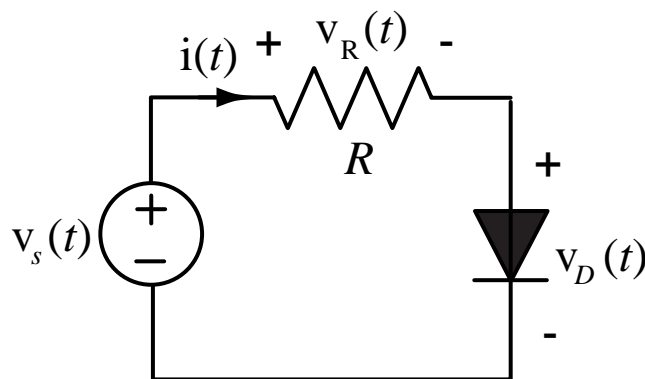

Figure C-1 Diode circuit to demonstrate linear analysis of a non-linear system

By Kirchhoff's voltage law the voltage across the diode is equal to the voltage across the series combination of the voltage source and the resistor,

$$v_D(t) = v_s(t) - R\,i(t) \quad . \tag{C.1}$$

Equation (C.1) can be solved for the current in the form,

$$i(t) = \frac{v_s(t) - v_D(t)}{R} \quad . \tag{C.2}$$

The current is also described by the diode equation,

$$i(t) = I_s\left(e^{q\,v_D(t)/kT} - 1\right) \quad . \tag{C.3}$$

This is a system of two equations, one of which is non-linear. The equations can be solved graphically by drawing the two V-I diagrams and finding the intersection (Figure C-2).

Diode Equation

$i(t)$

Kirchhoff's Voltage Law

$i(t)$

Slope $= -\dfrac{1}{R}$

$v_D(t)$

$v_s(t)$

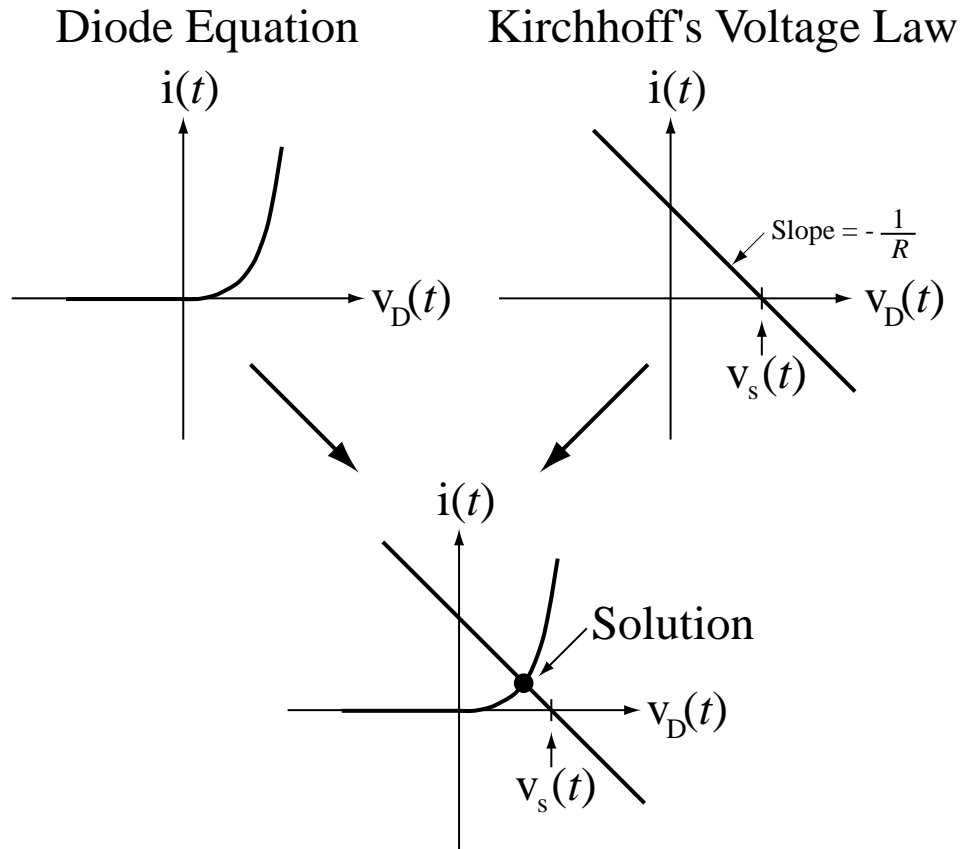$v_D(t)$

$i(t)$

Solution

$v_D(t)$

$v_s(t)$

Figure C-2 Graphical solution of the two diode-circuit equations

The solution in Figure C-2 is drawn as though the voltage and current $v_s(t)$ and $i(t)$ are constants. Suppose $v_s(t)$ consists of a constant $v_{s,DC}$ plus a small time-varying part $v_{s,AC}(t)$,

$$v_s(t) = v_{s,DC} + v_{s,AC}(t) \quad . \tag{C.4}$$

Then the graphical solution for the voltage and current would track the point of intersection of the two curves as $v_s(t)$ changes with time (Figure C-3).
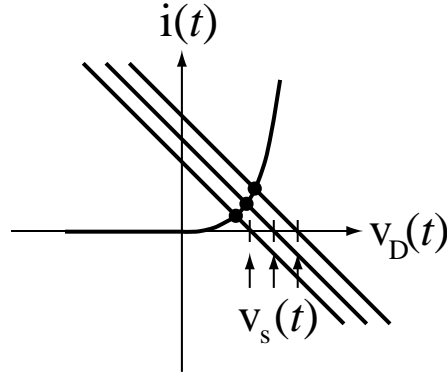
Figure C-3 Graphical solution of the diode circuit equations as the source voltage changes with time

If the time-varying part of the source voltage is small enough, the part of the diode curve traversed by the intersection point is very short and almost linear. Equation (C.1) can be re-written in a form that illustrates how to find an approximate solution analytically.

$$v_D(t) = v_{s,DC} + v_{s,AC}(t) - Ri(t) \tag{C.5}$$

From the diode equation,

$$i(t) = I_s\left(e^{q\,v_D(t)/kT} - 1\right) \ . \tag{C.6}$$

Combining (C.5) and (C.6),

$$v_D(t) = v_{s,DC} + v_{s,AC}(t) - RI_s\left(e^{q\,v_D(t)/kT} - 1\right) \ . \tag{C.7}$$

From the graphical solution we see that the diode voltage also consists of a constant plus a small variation,

$$v_D(t) = v_{D,DC} + v_{D,AC}(t) \ . \tag{C.8}$$

Therefore

$$v_D(t) = v_{s,DC} + v_{s,AC}(t) - RI_s\left(e^{q\left(v_{D,DC} + v_{D,AC}(t)\right)/kT} - 1\right) \tag{C.9}$$

or

$$v_D(t) = v_{s,DC} + v_{s,AC}(t) - RI_s\left(e^{q\,v_{D,DC}/kT}\,e^{q\,v_{D,AC}(t)/kT} - 1\right) \tag{C.10}$$

We can now linearize (C.10) by first expressing the exponential function in its series form,

$$e^{\frac{q\,v_{D,AC}(t)}{kT}} = 1 + \frac{q\,v_{D,AC}(t)}{kT} + \frac{1}{2}\left(\frac{q\,v_{D,AC}(t)}{kT}\right)^2 + \frac{1}{6}\left(\frac{q\,v_{D,AC}(t)}{kT}\right)^3 + \cdots \tag{C.11}$$

and then, assuming that the variation of the diode voltage is small enough, approximating the exponential by the first two terms of the series,

$$e^{q\mathrm{v}_{D,AC}(t)/kT} \cong 1 + \frac{q\mathrm{v}_{D,AC}(t)}{kT} \quad . \tag{C.12}$$

Then (C.10) becomes

$$\mathrm{v}_D(t) \cong \mathrm{v}_{s,DC} + \mathrm{v}_{s,AC}(t) - RI_s\left[e^{q\mathrm{v}_{D,DC}/kT}\left(1 + \frac{q\mathrm{v}_{D,AC}(t)}{kT}\right) - 1 \right] \tag{C.13}$$

or

$$\mathrm{v}_{D,DC} + \mathrm{v}_{D,AC}(t) \cong \mathrm{v}_{s,DC} - RI_s e^{q\mathrm{v}_{D,DC}/kT} + RI_s + \mathrm{v}_{s,AC}(t) - e^{q\mathrm{v}_{D,DC}/kT}\frac{qRI_s}{kT}\mathrm{v}_{D,AC}(t) \quad . \tag{C.14}$$

We can equate the constant parts and time-varying parts of (C.14) separately,

$$\mathrm{v}_{D,DC} \cong \mathrm{v}_{s,DC} - RI_s e^{q\mathrm{v}_{D,DC}/kT} + RI_s , \tag{C.15}$$

and

$$\mathrm{v}_{D,AC}(t) \cong \mathrm{v}_{s,AC}(t) - e^{q\mathrm{v}_{D,DC}/kT}\frac{qRI_s}{kT}\mathrm{v}_{D,AC}(t) \quad . \tag{C.16}$$

The equation for the constant part of the diode voltage (called the bias voltage) is non-linear and must still be solved by a graphical or numerical technique. Once the diode's bias voltage is found the second equation becomes a linear equation. That is, the equation for the time-varying part of the voltages and currents is approximately linear and can be solved by linear-system analysis techniques. So one non-linear dynamic system analysis has been converted into a static non-linear analysis plus an approximate linear dynamic analysis. This kind of technique is the basis for what is called *small-signal analysis* in electronics.