Web Appendix D - Derivations of Convolution Properties

D.1 Continuous-Time Convolution Properties

D.1.1 Commutativity Property

By making the change of variable, $\lambda = t - \tau$, in one form of the definition of CT convolution,

$$\mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau$$

it becomes

$$\mathrm{x}(t) * \mathrm{h}(t) = -\int_{\infty}^{-\infty} \mathrm{x}(t-\lambda)\mathrm{h}(\lambda)\,d\lambda = \int_{-\infty}^{\infty} \mathrm{h}(\lambda)\mathrm{x}(t-\lambda)\,d\lambda = \mathrm{h}(t) * \mathrm{x}(t)$$

proving that convolution is commutative.

D.1.2 Associativity Property

Associativity can be proven by considering the two operations

$$\left[\mathrm{x}(t) * \mathrm{y}(t)\right] * \mathrm{z}(t) \text{ and } \mathrm{x}(t) * \left[\mathrm{y}(t) * \mathrm{z}(t)\right].$$

Using the definition of convolution

$$\mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau$$

we get

$$\left[\mathrm{x}(t) * \mathrm{y}(t)\right] * \mathrm{z}(t) = \left[\int_{-\infty}^{\infty} \mathrm{x}(\tau_{xy})\mathrm{y}(t-\tau_{xy})\,d\tau_{xy}\right] * \mathrm{z}(t)$$

or

$$\left[\mathrm{x}(t) * \mathrm{y}(t)\right] * \mathrm{z}(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \mathrm{x}(\tau_{xy})\mathrm{y}(\tau_{yz}-\tau_{xy})\,d\tau_{xy}\right]\mathrm{z}(t-\tau_{yz})\,d\tau_{yz}$$

and

$$\mathrm{x}(t)*\left[\mathrm{y}(t)*\mathrm{z}(t)\right]=\mathrm{x}(t)*\left[\int_{-\infty}^{\infty}\mathrm{y}(\tau_{\mathrm{yz}})\mathrm{z}(t-\tau_{\mathrm{yz}})d\tau_{\mathrm{yz}}\right]$$

or

$$\mathrm{x}(t)*\left[\mathrm{y}(t)*\mathrm{z}(t)\right]=\int_{-\infty}^{\infty}\mathrm{x}(\tau_{\mathrm{xy}})\left[\int_{-\infty}^{\infty}\mathrm{y}(\tau_{\mathrm{yz}})\mathrm{z}(t-\tau_{\mathrm{xy}}-\tau_{\mathrm{yz}})d\tau_{\mathrm{yz}}\right]d\tau_{\mathrm{xy}}.$$

Then the proof consists of showing that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{x}(\tau_{\mathrm{xy}})\mathrm{y}(\tau_{\mathrm{yz}}-\tau_{\mathrm{xy}})\mathrm{z}(t-\tau_{\mathrm{yz}})d\tau_{\mathrm{xy}}d\tau_{\mathrm{yz}}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{x}(\tau_{\mathrm{xy}})\mathrm{y}(\tau_{\mathrm{yz}})\mathrm{z}(t-\tau_{\mathrm{xy}}-\tau_{\mathrm{yz}})d\tau_{\mathrm{xy}}d\tau_{\mathrm{yz}}$$

In the right-hand τ_{xy} integration make the change of variable $\lambda=\tau_{\mathrm{xy}}+\tau_{\mathrm{yz}}$ and $d\lambda=d\tau_{\mathrm{xy}}$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{x}(\tau_{\mathrm{xy}})\mathrm{y}(\tau_{\mathrm{yz}}-\tau_{\mathrm{xy}})\mathrm{z}(t-\tau_{\mathrm{yz}})d\tau_{\mathrm{xy}}d\tau_{\mathrm{yz}}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{x}(\lambda-\tau_{\mathrm{yz}})\mathrm{y}(\tau_{\mathrm{yz}})\mathrm{z}(t-\lambda)d\lambda d\tau_{\mathrm{yz}}$$

Next, in the right-hand τ_{xy} integration make the change of variable $\eta=\lambda-\tau_{\mathrm{yz}}$ and $d\eta=-d\tau_{\mathrm{yz}}$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{x}(\tau_{\mathrm{xy}})\mathrm{y}(\tau_{\mathrm{yz}}-\tau_{\mathrm{xy}})\mathrm{z}(t-\tau_{\mathrm{yz}})d\tau_{\mathrm{xy}}d\tau_{\mathrm{yz}}=-\int_{-\infty}^{\infty}\int_{\infty}^{-\infty}\mathrm{x}(\eta)\mathrm{y}(\lambda-\eta)\mathrm{z}(t-\lambda)d\lambda d\eta \qquad \text{(D.1)}$$

or

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{x}(\tau_{\mathrm{xy}})\mathrm{y}(\tau_{\mathrm{yz}}-\tau_{\mathrm{xy}})\mathrm{z}(t-\tau_{\mathrm{yz}})d\tau_{\mathrm{xy}}d\tau_{\mathrm{yz}}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{x}(\eta)\mathrm{y}(\lambda-\eta)\mathrm{z}(t-\lambda)d\lambda d\eta \qquad \text{(D.2)}$$

Except for the names of the variables of integration, the two integrals (D.1) and (D.2) are the same, therefore the integrals are equal and the associativity of convolution is proven.

D.1.3 Distributivity Property

Convolution is also distributive,

$$\mathrm{x}(t)*\left[\mathrm{h}_1(t)+\mathrm{h}_2(t)\right]=\mathrm{x}(t)*\mathrm{h}_1(t)+\mathrm{x}(t)*\mathrm{h}_2(t).$$

$$\mathrm{x}(t)*\left[\mathrm{h}_1(t)+\mathrm{h}_2(t)\right]=\int_{-\infty}^{\infty}\mathrm{x}(t)\left[\mathrm{h}_1(t-\tau)+\mathrm{h}_2(t-\tau)\right]d\tau$$

$$\mathrm{x}(t) * \left[\mathrm{h}_1(t) + \mathrm{h}_2(t)\right] = \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{h}_1(t-\tau)\,d\tau + \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{h}_2(t-\tau)\,d\tau$$

$$\mathrm{x}(t) * \left[\mathrm{h}_1(t) + \mathrm{h}_2(t)\right] = \mathrm{x}(t) * \mathrm{h}_1(t) + \mathrm{x}(t) * \mathrm{h}_2(t)$$

D.1.4 Differentiation Property

Let $\mathrm{y}(t)$ be the convolution of $\mathrm{x}(t)$ with $\mathrm{h}(t)$

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau .$$

Taking the derivative of $\mathrm{y}(t)$ with respect to time,

$$\mathrm{y}'(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}'(t-\tau)\,d\tau = \mathrm{x}(t) * \mathrm{h}'(t)$$

and, invoking the commutativity of convolution,

$$\mathrm{y}'(t) = \mathrm{x}'(t) * \mathrm{h}(t).$$

D.1.5 Area Property

Let $\mathrm{y}(t)$ be the convolution of $\mathrm{x}(t)$ with $\mathrm{h}(t)$

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau .$$

The area under $\mathrm{y}(t)$ is

$$\int_{-\infty}^{\infty} \mathrm{y}(t)\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau dt$$

or, exchanging the order of integration,

$$\int_{-\infty}^{\infty} \mathrm{y}(t)\,dt = \underbrace{\int_{-\infty}^{\infty} \mathrm{x}(\tau)\,d\tau}_{\text{Area of x}} \underbrace{\int_{-\infty}^{\infty} \mathrm{h}(t-\tau)\,dt}_{\text{Area of h}}$$

proving that the area of y is the product of the areas of x and h.

D.1.6 Scaling Property

Let $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ and $\mathrm{z}(t) = \mathrm{x}(at) * \mathrm{h}(at)$, $a > 0$. Then

$$\mathrm{z}(t) = \int_{-\infty}^{\infty} \mathrm{x}(a\tau)\mathrm{h}(a(t-\tau))\,d\tau .$$

Making the change of variable, $\lambda = a\tau \Rightarrow d\tau = d\lambda / a$, for $a > 0$ we get

$$\mathrm{z}(t) = \frac{1}{a}\int_{-\infty}^{\infty} \mathrm{x}(\lambda)\mathrm{h}(at-\lambda)\,d\lambda .$$

Since

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau$$

it follows that $\mathrm{z}(t) = (1/a)\mathrm{y}(at)$ and $(1/a)\mathrm{y}(at) = \mathrm{x}(at) * \mathrm{h}(at)$. If we do a similar proof for $a < 0$ we get $-(1/a)\mathrm{y}(at) = \mathrm{x}(at) * \mathrm{h}(at)$. Therefore, in general, if $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ then

$$\mathrm{y}(at) = |a|\,\mathrm{x}(at) * \mathrm{h}(at).$$

D.2 Discrete-Time Convolution Properties

D.2.1 Commutativity Property

The commutativity of DT convolution can be proven by starting with the definition of convolution

$$\mathrm{x}[n] * \mathrm{h}[n] = \sum_{k=-\infty}^{\infty} \mathrm{x}[k]\mathrm{h}[n-k]$$

and letting $q = n - k$. Then we have

$$\mathrm{x}[n] * \mathrm{h}[n] = \sum_{q=-\infty}^{\infty} \mathrm{x}[n-q]\mathrm{h}[q] = \sum_{q=-\infty}^{\infty} \mathrm{h}[q]\mathrm{x}[n-q] = \mathrm{h}[n] * \mathrm{x}[n]$$

D.2.2 Associativity Property

If we convolve $\mathrm{g}[n] = \mathrm{x}[n] * \mathrm{y}[n]$ with $\mathrm{z}[n]$ we get

$$\mathrm{g}[n] * \mathrm{z}[n] = \left(\mathrm{x}[n] * \mathrm{y}[n]\right) * \mathrm{z}[n] = \underbrace{\left(\sum_{k=-\infty}^{\infty} \mathrm{x}[k]\mathrm{y}[n-k]\right)}_{\mathrm{g}[n]} * \mathrm{z}[n]$$

or

$$\mathrm{g}[n] * \mathrm{z}[n] = \sum_{q=-\infty}^{\infty} \underbrace{\left(\sum_{k=-\infty}^{\infty} \mathrm{x}[k]\mathrm{y}[q-k]\right)}_{\mathrm{g}[q]} \mathrm{z}[n-q]$$

Exchanging the order of summation,

$$\left(\mathrm{x}[n] * \mathrm{y}[n]\right) * \mathrm{z}[n] = \sum_{k=-\infty}^{\infty} \mathrm{x}[k] \sum_{q=-\infty}^{\infty} \mathrm{y}[q-k]\mathrm{z}[n-q]$$

Let $n - q = m$ and let $\mathrm{h}[n] = \mathrm{y}[n] * \mathrm{z}[n]$. Then

$$\left(\mathrm{x}[n] * \mathrm{y}[n]\right) * \mathrm{z}[n] = \sum_{k=-\infty}^{\infty} \mathrm{x}[k] \underbrace{\sum_{m=-\infty}^{\infty} \mathrm{z}[m]\mathrm{y}[(n-k)-m]}_{\mathrm{z}[n]*\mathrm{y}[n-k]=\mathrm{y}[n-k]*\mathrm{z}[n]=\mathrm{h}[n-k]}$$

or

$$\left(\mathrm{x}[n] * \mathrm{y}[n]\right) * \mathrm{z}[n] = \underbrace{\sum_{k=-\infty}^{\infty} \mathrm{x}[k]\mathrm{h}[n-k]}_{\mathrm{x}[n]*\mathrm{h}[n]} = \mathrm{x}[n] * \left(\underbrace{\mathrm{y}[n] * \mathrm{z}[n]}_{\mathrm{h}[n]}\right)$$

D.2.3 Distributivity Property

If we convolve $\mathrm{x}[n]$ with the sum of $\mathrm{y}[n]$ and $\mathrm{z}[n]$ we get

$$\mathrm{x}[n] * \left(\mathrm{y}[n] + \mathrm{z}[n]\right) = \sum_{k=-\infty}^{\infty} \mathrm{x}[k]\left(\mathrm{y}[n-k] + \mathrm{z}[n-k]\right)$$

or

$$\mathrm{x}[n] * \left(\mathrm{y}[n] + \mathrm{z}[n]\right) = \underbrace{\sum_{k=-\infty}^{\infty} \mathrm{x}[k]\mathrm{y}[n-k]}_{=\mathrm{x}[n]*\mathrm{y}[n]} + \underbrace{\sum_{k=-\infty}^{\infty} \mathrm{x}[k]\mathrm{z}[n-k]}_{=\mathrm{x}[n]*\mathrm{z}[n]} .$$

Therefore

$$\mathrm{x}[n] * \left(\mathrm{y}[n] + \mathrm{z}[n]\right) = \mathrm{x}[n] * \mathrm{y}[n] + \mathrm{x}[n] * \mathrm{z}[n].$$

D.2.4 Differencing Property

Let $\mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n]$. Using the time-shifting property

$$\mathrm{y}[n-n_0] = \mathrm{x}[n] * \mathrm{h}[n-n_0] = \mathrm{x}[n-n_0] * \mathrm{h}[n]$$

the first backward difference of their convolution sum is

$$\mathrm{y}[n] - \mathrm{y}[n-1] = \mathrm{x}[n] * \mathrm{h}[n] - \mathrm{x}[n] * \mathrm{h}[n-1]$$

or

$$\mathrm{y}[n] - \mathrm{y}[n-1] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m] - \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m-1].$$

Combining summations,

$$\mathrm{y}[n] - \mathrm{y}[n-1] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\left(\mathrm{h}[n-m] - \mathrm{h}[n-m-1]\right)$$

or

$$\mathrm{y}[n] - \mathrm{y}[n-1] = \mathrm{x}[n] * \left(\mathrm{h}[n] - \mathrm{h}[n-1]\right)$$

D.2.5 Sum Property

Let $\mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n]$ and let the sum of all the impulses in the functions y, x, and h be S_{y}, S_{x} and S_h, respectively. Then

$$\mathrm{y}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m]$$

and

$$S_{\mathrm{y}} = \sum_{n=-\infty}^{\infty} \mathrm{y}[n] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m].$$

Interchanging the order of summation,

$$S_{\mathrm{y}} = \underbrace{\sum_{m=-\infty}^{\infty} \mathrm{x}[m]}_{=S_{\mathrm{x}}} \underbrace{\sum_{n=-\infty}^{\infty} \mathrm{h}[n-m]}_{=S_h} = S_{\mathrm{x}} S_h.$$