

Appendix E - An Exploration of Impulse Properties using Convolution

An important property of convolution is that when we convolve any function $g(t)$ with a unit impulse $\delta(t)$ the result is just $g(t)$. Convolution of $\delta(t)$ with any function leaves it unchanged. This holds true for the impulse itself,

$$\delta(t) * \delta(t) = \delta(t). \quad (\text{E.1})$$

We can see that by the following argument. We earlier described the unit impulse as the limit of a unit-area rectangular pulse as the width approaches zero,

$$\delta(t) = \lim_{a \rightarrow 0} (1/a) \text{rect}(t/a). \quad (\text{E.2})$$

If we convolve $(1/a) \text{rect}(t/a)$ with itself we get,

$$(1/a) \text{rect}(t/a) * (1/a) \text{rect}(t/a) = (1/a) \text{tri}(t/a) \quad (\text{E.3})$$

(Figure E-1).

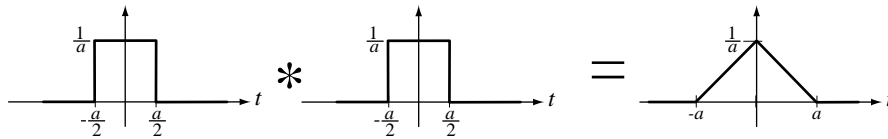


Figure E-1 Convolution of a rectangle with itself

If we now take the limit of both sides of (E.3) as a approaches zero,

$$\lim_{a \rightarrow 0} [(1/a) \text{rect}(t/a) * (1/a) \text{rect}(t/a)] = \lim_{a \rightarrow 0} (1/a) \text{tri}(t/a). \quad (\text{E.4})$$

From (E.1) and (E.2) the left side of (E.4) must be $\delta(t) * \delta(t)$. In Chapter 2 we showed that

$$\delta(t) = \lim_{a \rightarrow 0} (1/a) \text{tri}(t/a).$$

Therefore $\delta(t) * \delta(t) = \delta(t)$.

As demonstrated in Chapter 2 when the unit impulse was defined, this shows that a unit impulse can be conceived as the limit of a unit-area rectangle as the width approaches zero or as the limit of a unit-area triangle as the width approaches zero. In the limit the two functions are equivalent. This is one of the qualities that makes the impulse different

from normal functions. We could carry the argument further to show that a unit triangle convolved with itself is also an impulse in that same limit. This result re-emphasizes an important concept, the shape of an impulse is not important, only its strength. In fact, as these functions approach zero width, the whole concept of shape loses its meaning. There is no time in which to develop a shape. Other functions also approach a unit impulse as their widths approach zero. For example

$$\delta(t) = \lim_{a \rightarrow 0} \left(\frac{1}{a} \right) \text{sinc} \left(\frac{t}{a} \right) = \lim_{a \rightarrow 0} \frac{e^{t^2/2a^2}}{a\sqrt{2\pi}} .$$

An impulse can even be defined by

$$\delta(t) = \int_{-\infty}^{\infty} e^{-j2\pi ft} df$$

(Web Appendix H).

The fact that all these different-looking descriptions of a unit impulse are correct emphasizes the point that an impulse is different from an ordinary function. An ordinary function is defined by its value at every value of its independent variable. An impulse can be defined many ways. One way to define an impulse is by the effect it has when it is convolved with other functions. This leads back to a property of the unit impulse proven earlier $g(t) * \delta(t) = g(t)$. This is yet another way of defining an impulse. The other definitions follow from this one. For example, let $g(t) = 1$. Then

$$\delta(t) * g(t) = 1 = \int_{-\infty}^{\infty} \delta(\tau) \underbrace{g(t-\tau)}_{=1} d\tau = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

proving that the strength of a unit impulse is one.

Consider a system whose response is the derivative of its excitation $y(t) = x'(t)$. Then its impulse response is the derivative of the unit impulse which was earlier identified as the unit doublet $h(t) = u_1(t)$. For any general excitation, the response is the impulse response convolved with the excitation $y(t) = x'(t) = x(t) * u_1(t)$. So just as $g(t) * \delta(t) = g(t)$ can be used to define a unit impulse, $g(t) * u_1(t) = g'(t)$ can be used to define a unit doublet. This concept can be extended to all the singularity functions,

$$g(t) * u_2(t) = g''(t) , g(t) * u_3(t) = g'''(t) , \text{ etc...} \quad (\text{E.5})$$

Using the principle that the second derivative is the first derivative of the first derivative we can write

$$g''(t) = (g'(t))' = (g(t) * u_1(t))' = g(t) * u_1(t) * u_1(t)$$

and, using (E.5), $g(t) * u_2(t) = g(t) * u_1(t) * u_1(t)$. So it follows that $u_2(t) = u_1(t) * u_1(t)$. This can be extended to a general principle for the convolution of any of the singularity functions, $u_{m+n}(t) = u_m(t) * u_n(t)$. Applying this to the unit step we would get $u_{-1}(t) * u_{-1}(t) = u_{-2}(t) = \text{ramp}(t)$. From these properties we can form a general principle. If a system has a known impulse response $h(t)$ then the response to the singularity function $u_n(t)$ is the n th derivative of $h(t)$ and, if n is negative, that indicates that integration is to be done instead of differentiation.