Web Appendix F - Derivations of the Properties of the Continuous-Time Fourier Series

F.1 Numerical Computation of the CTFS

The harmonic function of a periodic signal with period T_F is

$$\mathrm{X}[k] = \frac{1}{T_F} \int_{T_F} \mathrm{x}(t) e^{-j2\pi k f_F t} dt .$$

Since the starting point of the integral is arbitrary, for convenience set it to $t = 0$

$$\mathrm{X}[k] = \frac{1}{T_F} \int_0^{T_F} \mathrm{x}(t) e^{-j2\pi k f_F t} dt .$$

Suppose we don't know the function $\mathrm{x}(t)$ but we have a set of N_F samples over one period starting at $t = 0$, the time between samples is $T_s = T_F / N_F$. Then we can approximate the integral by the sum of several integrals, each covering a time of length T_s

$$\mathrm{X}[k] \cong \frac{1}{T_F} \sum_{n=0}^{N_F - 1} \left[\int_{nT_s}^{(n+1)T_s} \mathrm{x}(nT_s) e^{-j2\pi k f_F nT_s} dt \right] \quad \text{(F.1)}$$

(Figure F-1).

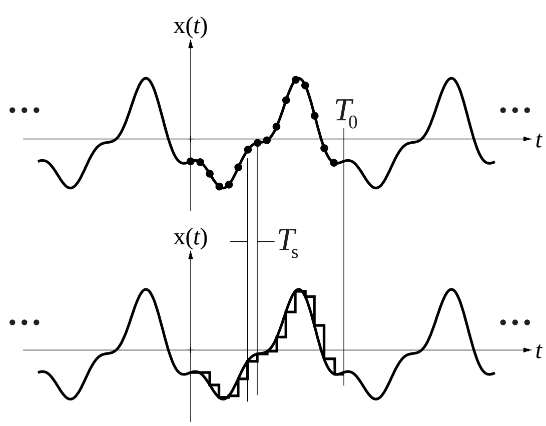

Figure F-1 Sampling the arbitrary periodic signal to estimate its CTFS harmonic function

(In Figure F-1, the samples extend over one fundamental period but they could extend over any period and the analysis would still be correct.) If the samples are close enough together $\mathrm{x}(t)$ does not change much between samples and the integral (F.1) becomes a good approximation. We can now complete the integration.

$$\mathrm{X}[k] \cong \frac{1}{T_F} \sum_{n=0}^{N_F-1} \left[\mathrm{x}(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-j2\pi k f_F t} dt \right] = \frac{1}{T_F} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) \left[\frac{e^{-j2\pi k f_F t}}{-j2\pi k f_F} \right]_{nT_s}^{(n+1)T_s}$$

$$\mathrm{X}[k] \cong \frac{1}{T_F} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) \left[\frac{e^{-j2\pi k f_F nT_s} - e^{-j2\pi k f_F (n+1)T_s}}{j2\pi k f_F} \right] = \frac{1 - e^{-j2\pi k f_F T_s}}{j2\pi k f_F T_F} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) e^{-j2\pi k f_F nT_s}$$

Using $T_s = T_F / N_F$,

$$\mathrm{X}[k] \cong \frac{1 - e^{-j2\pi k / N_F}}{j2\pi k} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) e^{-j2\pi nk / N_F} = e^{-j\pi k / N_F} \frac{\overbrace{e^{j\pi k / N_F} - e^{-j\pi k / N_F}}^{j2\sin(\pi k / N_F)}}{j2\pi k} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) e^{-j2\pi nk / N_F}$$

$$\mathrm{X}[k] \cong e^{-j\pi k / N_F} \frac{1}{N_F} \underbrace{\frac{\sin(\pi k / N_F)}{\pi k / N_F}}_{\mathrm{sinc}(k / N_F)} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) e^{-j2\pi nk / N_F}$$

$$\cong e^{-j\pi k / N_F} \frac{\mathrm{sinc}(k / N_F)}{N_F} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) e^{-j2\pi nk / N_F}$$

For harmonic numbers $|k| << N_F$ we can further approximate the harmonic function as

$$\mathrm{X}[k] \cong \frac{1}{N_F} \sum_{n=0}^{N_F-1} \mathrm{x}(nT_s) e^{-j2\pi nk / N_F} .$$

F.2 Linearity

Let $\mathrm{z}(t) = \alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t)$. If $T_F = mT_{0x} = qT_{0y}$, where m and q are integers, then the CTFS harmonic function of $\mathrm{z}(t)$ is

$$\mathrm{Z}[k] = \frac{1}{T_F} \int_{T_F} \mathrm{z}(t) e^{-j2\pi k f_F t} dt = \frac{1}{T_F} \int_{T_F} \left[\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \right] e^{-j2\pi k f_F t} dt$$

and

$$\mathrm{z}(t)=\sum_{n=-\infty}^{\infty}\mathrm{Z}[k]e^{j2\pi kf_F t}=\sum_{n=-\infty}^{\infty}\left(\alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]\right)e^{j2\pi kf_F t}$$

and

$$\mathrm{Z}[k]=\alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]\ .$$

So the linearity property is

$$T_F=mT_{0x}=qT_{0y}$$
$$\alpha\,\mathrm{x}(t)+\beta\,\mathrm{y}(t)\xleftrightarrow{\mathrm{FS}}\alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]\ .$$

F.3 Time Shifting

Let $\mathrm{z}(t)=\mathrm{x}(t-t_0)$ and let $T_F=mT_{0x}=mT_{0z}$, where m is an integer. Then

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\mathrm{z}(t)e^{-jk\omega_F t}dt=\frac{1}{T_F}\int_{T_F}\mathrm{x}(t-t_0)e^{-jk\omega_F t}dt\ .$$

Making the change of variable $\tau=t-t_0\Rightarrow d\tau=dt$,

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\mathrm{x}(\tau)e^{-jk\omega_F(\tau+t_0)}d\tau=e^{-jk\omega_F t_0}\underbrace{\frac{1}{T_F}\int_{T_F}\mathrm{x}(\tau)e^{-jk\omega_F \tau}d\tau}_{=\mathrm{X}[k]}$$
$$\mathrm{Z}[k]=e^{-jk\omega_F t_0}\,\mathrm{X}[k]\ .$$

So the time-shifting property is

$$T_F=mT_0$$
$$\mathrm{x}(t-t_0)\xleftrightarrow{\mathrm{FS}}e^{-j2\pi kf_F t_0}\,\mathrm{X}[k]\ .$$
$$\mathrm{x}(t-t_0)\xleftrightarrow{\mathrm{FS}}e^{-jk\omega_F t_0}\,\mathrm{X}[k]$$

F.4 Frequency Shifting

Let $\mathrm{z}(t)=e^{j2\pi k_0 f_F t}\,\mathrm{x}(t)$, with k_0 being an integer and $T_F = mT_{0x} = mT_{0z}$, where m is an integer. Then

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\mathrm{z}(t)e^{-j2\pi k f_F t}dt=\frac{1}{T_F}\int_{T_F}e^{j2\pi k_0 f_F t}\,\mathrm{x}(t)e^{-j2\pi k f_F t}dt$$

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\mathrm{x}(t)e^{-j2\pi(k-k_0)f_F t}dt=\mathrm{X}[k-k_0].$$

So the frequency shifting property is

$$\begin{gathered} T_F = mT_0 \\ e^{j2\pi k_0 f_F t}\,\mathrm{x}(t)\xleftrightarrow{\mathcal{FS}}\mathrm{X}[k-k_0]\ . \\ e^{jk_0\omega_F t}\,\mathrm{x}(t)\xleftrightarrow{\mathcal{FS}}\mathrm{X}[k-k_0] \end{gathered}$$

F.5 Time Reversal

Let $\mathrm{z}(t)=\mathrm{x}(-t)$ and let $T_F = mT_{0x} = mT_{0z}$, where m is an integer. If

$$\mathrm{x}(t)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{j2\pi k f_F t}$$

then

$$\mathrm{x}(-t)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{j2\pi k f_F(-t)}=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{j2\pi(-k f_F)t}\ .$$

Let $q=-k$, then

$$\mathrm{x}(-t)=\sum_{q=\infty}^{-\infty}\mathrm{X}[-q]e^{j2\pi q f_F t}$$

and, since changing the order of summation does not change the sum,

$$\mathrm{x}(-t)=\sum_{q=-\infty}^{\infty}\mathrm{X}[-q]e^{j2\pi q f_F t}\ .$$

Therefore, since

$$\mathrm{z}(t)=\sum_{k=-\infty}^{\infty}\mathrm{Z}[k]e^{j2\pi k f_F t}$$

we can say that

$$\sum_{k=-\infty}^{\infty} Z[k] e^{j2\pi k f_F t} = \sum_{k=-\infty}^{\infty} X[-k] e^{j2\pi k f_F t}$$

and $Z[k] = X[-k]$. So the time reversal property is

$$x(-t) \overset{T_F = mT_0}{\underset{}{\xleftrightarrow{FS}}} X[-k] \ .$$

F.6 Time Scaling

Let $z(t) = x(at)$, $a > 0$ and let $T_F = mT_{0x}$, m an integer. (Figure F-2).

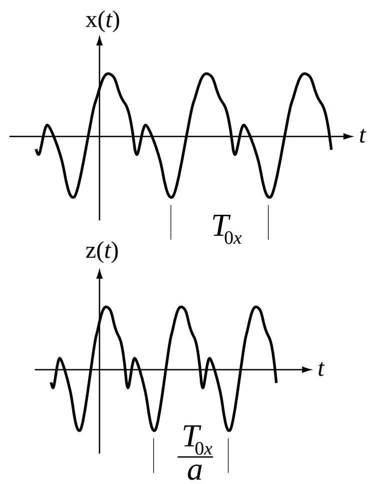

Figure F-2 A signal $x(t)$ and a time-scaled version $z(t)$ of that signal

The first thing to realize is that if $x(t)$ is periodic with fundamental period T_{0x} that $z(t)$ is periodic with fundamental period $T_{0z} = T_{0x} / a$ and fundamental frequency af_{0x}.

Case 1. $z(t)$ represented by a CTFS over a period of $z(t)$, T_F / a.

The CTFS harmonic function will be

$$Z[k] = \frac{a}{T_F} \int_{t_0}^{t_0 + T_F / a} z(t) e^{-j2\pi k a f_F t} dt = \frac{a}{T_F} \int_{t_0}^{t_0 + T_F / a} x(at) e^{-j2\pi k a f_F t} dt \ .$$

We can make the change of variable $\tau = at \Rightarrow d\tau = adt$ yielding

$$Z[k] = \frac{a}{T_F}\frac{1}{a}\int_{at_0}^{at_0+T_F} x(\tau)e^{-j2\pi kaf_F\tau/a}d\tau = \frac{1}{T_F}\int_{at_0}^{at_0+T_F} x(\tau)e^{-j2\pi kf_F\tau}d\tau .$$

Since the starting point t_0 is arbitrary

$$Z[k] = \frac{1}{T_F}\int_{T_F} x(\tau)e^{-j2\pi kf_F\tau}d\tau = X[k]$$

and the CTFS harmonic function describing $z(t)$ over the period T_F / a is the same as the CTFS harmonic function describing $x(t)$over the period T_F.

$$z(t) = x(at)$$
$$T_F = mT_{0x} \rightarrow T_F / a = mT_{0x}/\text{a}$$
$$Z[k] = X[k]$$

Even though the CTFS harmonic functions of $x(t)$ and $z(t)$ are the same, the CTFS representations themselves are not because the fundamental frequencies are different. The representations are

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j2\pi kf_F t} \text{ and } \quad z(t) = x(at) = \sum_{k=-\infty}^{\infty} Z[k]e^{j2\pi kaf_F t} \quad .$$

Case 2. $z(t)$ represented by a CTFS over a period of $x(t)$, T_F

The CTFS harmonic function will be

$$Z[k] = \frac{1}{T_F}\int_{t_0}^{t_0+T_F} z(t)e^{-j2\pi kf_F t}dt = \frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(at)e^{-j2\pi kf_F t}dt .$$

Let $\tau = at \Rightarrow d\tau = adt$. Then

$$Z[k] = \frac{1}{aT_F}\int_{at_0}^{at_0+aT_F} x(\tau)e^{-j2\pi kf_F\tau/a}d\tau .$$

If a is not an integer, the relationship between the two harmonic functions $Z[k]$ and $X[k]$ cannot be simplified further.

Let a be a non-zero integer. The signal $\mathrm{x}(t)$ is made up of frequency components at integer multiples of f_F. Therefore for ratios k/a that are not integers, $\mathrm{x}(\tau)$ and $e^{-j2\pi k f_F \tau/a}$ are orthogonal on the interval $at_0 < \tau < at_0 + aT_F$ and $\mathrm{Z}[k] = 0$. For ratios k/a that are integers, the integral over a periods is a times the integral over one period and

$$\mathrm{Z}[k] = a\left(\frac{1}{aT_F}\int_{at_0}^{at_0+T_F} \mathrm{x}(\tau)e^{-j2\pi(k/a)f_F\tau}d\tau\right) = \mathrm{X}[k/a] \ , \ k/a \text{ an integer}.$$

So the time scaling property, for this kind of time scaling, is

$$T_F = mT_{0x} \ , \ \mathrm{z}(t) = \mathrm{x}(at) \ , a \text{ a non-zero integer}$$
$$\mathrm{Z}[k] = \begin{cases} \mathrm{X}[k/a] & , \ k/a \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}.$$

F.7 Change of Period

If the CTFS harmonic function of $\mathrm{x}(t)$ over any period T_F is $\mathrm{X}[k]$, we can find the CTFS harmonic function $\mathrm{X}_q[k]$ of $\mathrm{x}(t)$ over a time qT_F where q is a positive integer. The new fundamental CTFS frequency is then f_F/q and

$$\mathrm{X}_q[k] = \frac{1}{qT_F}\int_{qT_F} \mathrm{x}(t)e^{-j2\pi(kf_F/q)t}dt$$

This is exactly the same as the result for time scaling by a positive integer in the previous section and the result is

$$T_F \to qT_F \Rightarrow \mathrm{X}_q[k] = \begin{cases} \mathrm{X}[k/q] & , \ k/q \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}.$$

F.8 Time Differentiation

Let $\mathrm{z}(t) = \frac{d}{dt}(\mathrm{x}(t))$ and let $T_F = mT_{0x} = mT_{0z}$, where m is an integer. Then we can represent $\mathrm{z}(t)$ by

$$z(t) = \frac{d}{dt}(x(t)) = \frac{d}{dt}\left(\sum_{n=-\infty}^{\infty} X[k] e^{jk\omega_F t}\right) = \sum_{n=-\infty}^{\infty} jk\omega_F X[k] e^{jk\omega_F t} .$$

Then, if

$$z(t) = \sum_{k=-\infty}^{\infty} Z[k] e^{jk\omega_F t}$$

it follows that

$$\sum_{k=-\infty}^{\infty} Z[k] e^{jk\omega_F t} = jk\omega_F \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_F t} ,$$

$Z[k] = jk\omega_F X[k]$.

So the differentiation property is

$$T_F = mT_0$$

$$\frac{d}{dt}(x(t)) \xleftrightarrow{\mathcal{FS}} j2\pi k f_F X[k] \;,\; \frac{d}{dt}(x(t)) \xleftrightarrow{\mathcal{FS}} jk\omega_F X[k] .$$

F.9 Time Integration

Let $z(t) = \int_{-\infty}^{t} x(\tau) d\tau$ and let $T_F = mT_{0x}$, where m is an integer. We must consider two cases separately, $X[0] = 0$ and $X[0] \neq 0$. If $X[0] \neq 0$ then, even though $x(t)$ is periodic, $z(t)$ is not and we cannot represent it exactly for all time with a CTFS (Figure F-3).

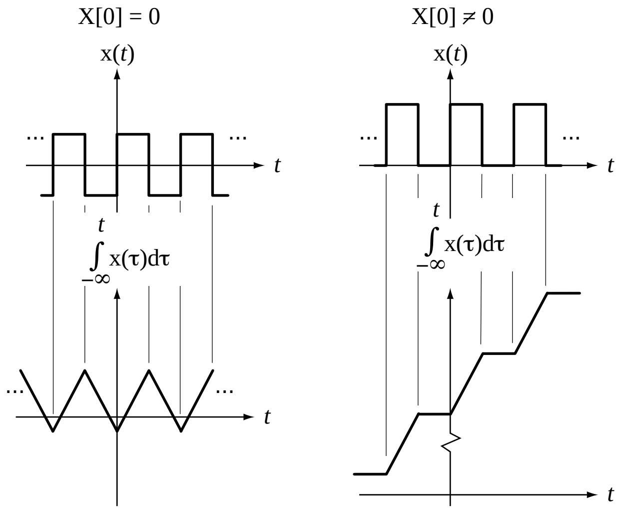

Figure F-3 Effect of a non-zero average value on the integral of a periodic function

If $\mathrm{X}[0]=0$ then we can represent $\mathrm{z}(t)$ by

$$\mathrm{z}(t)=\int_{-\infty}^{t}\mathrm{x}(\tau)\,d\tau$$

and its CTFS harmonic function is

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\left[\int_{-\infty}^{t}\mathrm{x}(\tau)\,d\tau\right]e^{-jk\omega_F t}\,dt\,.$$

For $k=0$,

$$\mathrm{Z}[0]=\frac{1}{T_F}\int_{T_F}\left[\int_{-\infty}^{t}\mathrm{x}(\tau)\,d\tau\right]dt$$

which is the average value of the integral of $\mathrm{x}(t)$ over one fundamental period. We know that the average value of $\mathrm{x}(t)$ is zero but, without some other information, we don't know what the average value of the integral of $\mathrm{x}(t)$ is and cannot determine the value of $\mathrm{Z}[0]$. However we <u>can</u> determine all the other values of $\mathrm{Z}[k]$.

$$\mathrm{z}(t)=\int_{-\infty}^{t}\mathrm{x}(\tau)\,d\tau=\int_{-\infty}^{t}\underbrace{\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{jk\omega_F\tau}}_{\mathrm{x}(\tau)}d\tau=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]\int_{-\infty}^{t}e^{jk\omega_F\tau}\,d\tau$$

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\left[\sum_{q=-\infty}^{\infty}\mathrm{X}[q]\int_{-\infty}^{t}e^{jq\omega_F\tau}\,d\tau\right]e^{-jk\omega_F t}\,dt \qquad \text{(F.2)}$$

In (F.2) k has been replaced by q to avoid confusion with k which is independent of q. To finish the integration we must evaluate the integral $\int_{-\infty}^{t} e^{jq\omega_F \tau} d\tau$ which is

$$\left[\frac{e^{jq\omega_F \tau}}{jq\omega_F}\right]_{-\infty}^{t} .$$

Evaluation at the upper limit is not a problem but the lower limit does present a problem. Since the complex sinusoid $e^{jq\omega_F \tau}$ is periodic it is impossible to define what its value is at the lower limit of negative infinity because negative infinity is a limit, not a number. The magnitude of $e^{jq\omega_F \tau}$ is one but its phase could be any value in a range of 2π radians. But, it turns out that this won't matter. Call the indeterminate value at the lower limit C. Then

$$\left[\frac{e^{jq\omega_F \tau}}{jq\omega_F}\right]_{-\infty}^{t} = \frac{e^{jq\omega_F t}}{jq\omega_F} - C$$

and

$$Z[k] = \frac{1}{T_F}\int_{T_F} \sum_{q=-\infty}^{\infty} \mathrm{X}[q]\left[\frac{e^{jq\omega_F t}}{jq\omega_F} - C\right] e^{-jk\omega_F t} dt$$
$$= \sum_{q=-\infty}^{\infty} \frac{\mathrm{X}[q]}{T_F}\int_{T_F}\left[\frac{e^{j(q-k)\omega_F t}}{jq\omega_F} - Ce^{-jk\omega_F t}\right] dt$$

For each $k \neq 0$ the computation involves a summation of terms involving q. But all of those terms are zero unless $q = k$ because the integration is over a integer number of periods of a complex sinusoid. So, in the end,

$$Z[k] = \frac{\mathrm{X}[k]}{T_F}\int_{T_F}\left[\frac{1}{jk\omega_F}\right] dt = \frac{\mathrm{X}[k]}{jk\omega_F T_F} T_F = \frac{\mathrm{X}[k]}{jk\omega_F} \quad , \quad k \neq 0$$

and the integration property is

$$T_F = mT_0$$
$$\int_{-\infty}^{t} \mathrm{x}(\tau)\, d\tau \xleftarrow{\mathcal{FS}} \frac{\mathrm{X}[k]}{j2\pi k f_F} \quad , \quad \int_{-\infty}^{t} \mathrm{x}(\tau)\, d\tau \xleftarrow{\mathcal{FS}} \frac{\mathrm{X}[k]}{jk\omega_F} \quad , \quad k \neq 0 \quad , \quad \text{if} \quad \mathrm{X}[0] = 0 \quad .$$

F.10 Multiplication-Convolution Duality

Let $\mathrm{z}(t)=\mathrm{x}(t)\mathrm{y}(t)$ and let $T_F = mT_{0x} = qT_{0y}$, where m and q are integers. Then

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\mathrm{z}(t)e^{-j2\pi k f_F t}dt=\frac{1}{T_F}\int_{T_F}\mathrm{x}(t)\mathrm{y}(t)e^{-j2\pi k f_F t}dt\ .$$

Then, using

$$\mathrm{y}(t)=\sum_{k=-\infty}^{\infty}\mathrm{Y}[k]e^{j2\pi k f_F t}=\sum_{q=-\infty}^{\infty}\mathrm{Y}[q]e^{j2\pi q f_F t}$$

we get

$$\mathrm{Z}[k]=\frac{1}{T_F}\int_{T_F}\mathrm{x}(t)\left(\sum_{q=-\infty}^{\infty}\mathrm{Y}[q]e^{j2\pi q f_F t}\right)e^{-j2\pi k f_F t}dt\ .$$

Reversing the order of integration and summation,

$$\mathrm{Z}[k]=\frac{1}{T_F}\sum_{q=-\infty}^{\infty}\mathrm{Y}[q]\int_{T_F}\mathrm{x}(t)e^{j2\pi q f_F t}e^{-j2\pi k f_F t}dt$$

or

$$\mathrm{Z}[k]=\sum_{q=-\infty}^{\infty}\mathrm{Y}[q]\underbrace{\frac{1}{T_F}\int_{T_F}\mathrm{x}(t)e^{-j2\pi(k-q)f_F t}dt}_{=\mathrm{X}[k-q]}\ .$$

Then

$$\mathrm{Z}[k]=\sum_{q=-\infty}^{\infty}\mathrm{Y}[q]\mathrm{X}[k-q]$$

and the multiplication property is

$$\mathrm{x}(t)\mathrm{y}(t)\xleftrightarrow[\;]{\mathcal{FS}}\sum_{q=-\infty}^{\infty}\mathrm{Y}[q]\mathrm{X}[k-q]=\mathrm{X}[k]*\mathrm{Y}[k] \qquad T_F = mT_{0x} = qT_{0y}\ .$$

This result $\sum_{q=-\infty}^{\infty}\mathrm{Y}[q]\mathrm{X}[k-q]$ is a *convolution sum*. So the product of CT signals corresponds to the convolution sum of their CTFS harmonic functions.

Now let $\mathrm{Z}[k]=\mathrm{X}[k]\mathrm{Y}[k]$ and let $T_F = mT_{0x} = qT_{0y}$, where m and q are integers. Then

$$z(t) = \sum_{k=-\infty}^{\infty} X[k] Y[k] e^{jk\omega_F t}$$

$$z(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_F} \int_{T_F} x(\tau) e^{-jk\omega_F \tau} d\tau \, Y[k] e^{jk\omega_F t} = \frac{1}{T_F} \int_{T_F} x(\tau) d\tau \underbrace{\sum_{k=-\infty}^{\infty} Y[k] e^{jk\omega_F (t-\tau)}}_{y(t-\tau)}$$

or

$$z(t) = \frac{1}{T_F} \int_{T_F} x(\tau) y(t-\tau) d\tau \ .$$

This integral looks just like a convolution integral except that it covers the range $t_0 \le \tau < t_0 + T_F$ instead of $-\infty < \tau < \infty$. This integral operation is called *periodic convolution* and is indicated by the notation

$$x(t) \circledast y(t) = \int_{T_F} x(\tau) y(t-\tau) d\tau \ .$$

Therefore

$$z(t) = (1/T_F) x(t) \circledast y(t) .$$

Since $x(t)$ is periodic it can be expressed as the periodic extension of an aperiodic function $x_{ap}(t)$

$$x(t) = \sum_{q=-\infty}^{\infty} x_{ap}(t - qT_F) = x_{ap}(t) * \delta_{T_F}(t) .$$

(The function, $x_{ap}(t)$, is not unique. It can be any function which satisfies this equation.) Then

$$x(t) \circledast y(t) = \int_{T_F} \left[\sum_{q=-\infty}^{\infty} x_{ap}(\tau - qT_F) \right] y(t-\tau) d\tau$$

$$x(t) \circledast y(t) = \sum_{q=-\infty}^{\infty} \int_{t_0}^{t_0+T_F} x_{ap}(\tau - qT_F) y(t-\tau) d\tau \ .$$

Let $\lambda = \tau - qT_F$. Then $d\lambda = d\tau$ and

$$x(t) \circledast y(t) = \sum_{q=-\infty}^{\infty} \int_{t_0+qT_F}^{t_0+(q+1)T_F} x_{ap}(\lambda) y(t - (\lambda + qT_F)) d\lambda .$$

Since $y(t)$ is periodic, with period T_F,

$$y\left(t-\left(\lambda+qT_F\right)\right)=y\left(t-qT_F-\lambda\right)=y\left(t-\lambda\right)$$

and the summation of integrals $\sum_{q=-\infty}^{\infty}\int_{t_0+qT_F}^{t_0+(q+1)T_F}$ is equivalent to the single integral over infinite limits $\int_{-\infty}^{\infty}$ we conclude that

$$x(t)\circledast y(t)=\int_{-\infty}^{\infty}x_{ap}(\lambda)y(t-\lambda)d\lambda=x_{ap}(t)*y(t) .$$

So the periodic convolution of two functions $x(t)$ and $y(t)$ each with period T_F can be expressed as an aperiodic convolution of $y(t)$ with a function $x_{ap}(t)$ which, when periodically repeated with the same period T_F equals $x(t)$. The periodic convolution of two periodic functions corresponds to the product of their CTFS harmonic function representations and the period T_F and the convolution property is

$$\begin{gathered}T_F=mT_{0x}=qT_{0y}\\ x(t)\circledast y(t)\xleftrightarrow{FS}T_F\,X[k]Y[k]\end{gathered} .$$

F.11 Conjugation

Let $z(t)=x^*(t)$ and let $T_F=mT_{0x}=mT_{0z}$, where m is an integer. Then

$$\sum_{k=-\infty}^{\infty}Z[k]e^{j2\pi kf_Ft}=\left(\sum_{k=-\infty}^{\infty}X[k]e^{j2\pi kf_Ft}\right)^*$$
$$=\sum_{k=-\infty}^{\infty}X^*[k]e^{-j2\pi kf_Ft}=\sum_{k=\infty}^{-\infty}X^*[-k]e^{j2\pi kf_Ft}$$

and, since changing the order of summation does not change the sum,

$$\sum_{k=-\infty}^{\infty}Z[k]e^{j2\pi kf_Ft}=\sum_{k=-\infty}^{\infty}X^*[-k]e^{j2\pi kf_Ft} ,$$

$Z[k]=X^*[-k]$ and the conjugation property is

$$\begin{gathered}T_F=mT_0\\ x^*(t)\xleftrightarrow{FS}X^*[-k]\end{gathered} .$$

F.12 Parseval's Theorem

The signal energy in any period $T_F = mT_{0x}$, where m is an integer, of any periodic signal $\mathrm{x}(t)$ is

$$E_{x,T_F} = \int_{T_F} \left|\mathrm{x}(t)\right|^2 dt = \int_{T_F} \left| \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{j2\pi k f_F t} \right|^2 dt$$

$$E_{x,T_F} = \int_{T_F} \left(\sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{j2\pi k f_F t} \right) \left(\sum_{q=-\infty}^{\infty} \mathrm{X}[q] e^{j2\pi q f_F t} \right)^* dt$$

$$E_{x,T_F} = \int_{T_F} \left(\sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathrm{X}[k] e^{j2\pi k f_F t} \mathrm{X}^*[q] e^{-j2\pi q f_F t} \right) dt$$

$$E_{x,T_F} = \int_{T_F} \left(\sum_{k=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathrm{X}[k] \mathrm{X}^*[q] e^{j2\pi(k-q) f_F t} \right) dt$$

$$E_{x,T_F} = \int_{T_F} \left(\sum_{k=-\infty}^{\infty} \mathrm{X}[k] \mathrm{X}^*[k] + \sum_{\substack{k=-\infty \\ k \neq q}}^{\infty} \sum_{q=-\infty}^{\infty} \mathrm{X}[k] \mathrm{X}^*[q] e^{j2\pi(k-q) f_F t} \right) dt$$

$$E_{x,T_F} = \int_{T_F} \sum_{k=-\infty}^{\infty} \left|\mathrm{X}[k]\right|^2 dt + \underbrace{\int_{T_F} \sum_{\substack{k=-\infty \\ k \neq q}}^{\infty} \sum_{q=-\infty}^{\infty} \mathrm{X}[k] \mathrm{X}^*[q] e^{j2\pi(k-q) f_F t} dt}_{=0,\, k \neq q}$$

$$E_{x,T_F} = T_F \sum_{k=-\infty}^{\infty} \left|\mathrm{X}[k]\right|^2$$

Therefore, for any periodic signal $\mathrm{x}(t)$

$$T_F = mT_0$$

$$\frac{1}{T_F} \int_{T_F} \left|\mathrm{x}(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|\mathrm{X}[k]\right|^2 \quad .$$