Web Appendix G - Derivations of the Properties of the Discrete-Time Fourier Series

G.1 The Trigonometric Discrete-Time Fourier Series

Assuming a DTFS harmonic function has been found, we can say that

$$\mathrm{x}[n] = \mathrm{x}_F[n] \ , \ n_0 < n < n_0 + N_F$$

where

$$\mathrm{x}_F[n] = \sum_{k=\langle N_F \rangle} \mathrm{X}[k] e^{j2\pi kn/N_F} \ . \qquad \text{(G.1)}$$

It is useful to explore the characteristics of the complex conjugate of $\mathrm{x}_F[n]$. If we conjugate both sides of (G.1) we get

$$\mathrm{x}_F^*[n] = \sum_{k=\langle N_F \rangle} \mathrm{X}^*[k] e^{-j2\pi kn/N_F} \ .$$

Since any range of k exactly N_F in length will work we can replace k with $-k$ and still have an equality

$$\mathrm{x}_F^*[n] = \sum_{k=\langle N_F \rangle} \mathrm{X}^*[-k] e^{j2\pi kn/N_F} \ .$$

In words, this says that to find the DTFS harmonic function $\mathrm{X}[k]$ for the complex conjugate of a signal, conjugate it and change the sign of k. The transformation is $\mathrm{X}[k] \rightarrow \mathrm{X}^*[-k]$ and then for any $\mathrm{x}[n]$, $\mathrm{x}^*[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}^*[-k]$. In the very important special case in which $\mathrm{x}[n]$ is a real-valued function, $\mathrm{x}[n] = \mathrm{x}^*[n]$ and therefore $\mathrm{x}_F[n] = \mathrm{x}_F^*[n]$. That means that the two representations,

$$\mathrm{x}_F[n] = \sum_{k=\langle N_F \rangle} \mathrm{X}[k] e^{j2\pi kn/N_F} \quad \text{and} \quad \mathrm{x}_F^*[n] = \sum_{k=\langle N_F \rangle} \mathrm{X}^*[-k] e^{j2\pi kn/N_F}$$

must be equal and therefore that $\mathrm{X}[k]=\mathrm{X}^*[-k]$, implying that, for real-valued signals and for any *k*, $\mathrm{X}[k]$ and $\mathrm{X}[-k]$ are complex conjugates.

Any set of consecutive harmonics exactly N_F in length is sufficient to represent a signal over the time range $n_0 \le n < n_0 + N_F$. The time range N_F is either an even integer or an odd integer. For N_F even, consider the harmonics $-N_F/2 \le k < N_F/2$. All the harmonics except $k=0$ and $k=-N_F/2$ occur in complex conjugate pairs $\pm 1, \pm 2, \cdots \pm (N_F/2-1)$. The $k=0$ harmonic is

$$\mathrm{X}[0]=\frac{1}{N_F}\sum_{n=n_0}^{n_0+N_F-1}\mathrm{x}[n]$$

and is therefore guaranteed to be a real number (if $\mathrm{x}[n]$ is real). The $k=-N_F/2$ harmonic is

$$\mathrm{X}[-N_F/2]=\frac{1}{N_F}\sum_{n=n_0}^{n_0+N_F-1}\mathrm{x}[n]e^{j\pi n}=\frac{1}{N_F}\sum_{n=n_0}^{n_0+N_F-1}\mathrm{x}[n]\cos(\pi n)$$
$$=\frac{1}{N_F}\sum_{n=n_0}^{n_0+N_F-1}\mathrm{x}[n](-1)^n$$

which is also guaranteed real. Therefore, we can write the DTFS representation of the signal, $\mathrm{x}_F[n]=\sum_{k=-N_F/2}^{N_F/2-1}\mathrm{X}[k]e^{j2\pi kn/N_F}$, as

$$\mathrm{x}_F[n]=\mathrm{X}[0]+\underbrace{\mathrm{X}[-N_F/2]}_{=\mathrm{X}[N_F/2]}\underbrace{\cos(\pi n)}_{=(-1)^n}+\sum_{k=1}^{N_F/2-1}\left[\mathrm{X}[k]e^{j2\pi kn/N_F}+\mathrm{X}^*[k]e^{-j2\pi kn/N_F}\right]$$

or

$$\mathrm{x}_F[n]=\mathrm{X}[0]+(-1)^n\mathrm{X}[N_F/2]$$
$$+\sum_{k=1}^{N_F/2-1}\left[\begin{array}{l}\mathrm{Re}(\mathrm{X}[k])e^{j2\pi kn/N_F}+\mathrm{Re}(\mathrm{X}^*[k])e^{-j2\pi kn/N_F}\\+j\,\mathrm{Im}(\mathrm{X}[k])e^{j2\pi kn/N_F}-j\,\mathrm{Im}(\mathrm{X}^*[k])e^{-j2\pi kn/N_F}\end{array}\right]$$

$$\mathrm{x}_F[n]=\mathrm{X}[0]+(-1)^n\mathrm{X}[N_F/2]+\sum_{k=1}^{N_F/2-1}\left[\begin{array}{l}2\,\mathrm{Re}(\mathrm{X}[k])\cos(2\pi kn/N_F)\\-2\,\mathrm{Im}(\mathrm{X}[k])\sin(2\pi kn/N_F)\end{array}\right]$$

$$x_F[n] = X[0] + (-1)^n X[N_F/2] + \sum_{k=1}^{N_F/2-1} \begin{bmatrix} (X[k] + X^*[k])\cos(2\pi kn/N_F) \\ + j(X[k] - X^*[k])\sin(2\pi kn/N_F) \end{bmatrix}$$

$$x_F[n] = X[0] + (-1)^n X[N_F/2] + \sum_{k=1}^{N_F/2-1} \left[X_c[k]\cos(2\pi kn/N_F) + X_s[k]\sin(2\pi kn/N_F) \right] \quad \text{(G.2)}$$

where $X_c[k] = X[k] + X^*[k]$ and $X_s[k] = j(X[k] - X^*[k])$, $0 < k < N_F/2$.

Therefore, using $X[k] = \frac{1}{N_F}\sum_{n=n_0}^{n_0+N_F-1} x[n]e^{-j2\pi kn/N_F}$,

$$X_c[k] = \frac{2}{N_F}\sum_{n=n_0}^{n_0+N_F-1} x[n]\cos(2\pi kn/N_F) \text{ and } X_s[k] = \frac{2}{N_F}\sum_{n=n_0}^{n_0+N_F-1} x[n]\sin(2\pi kn/N_F)$$

For N_F odd the development is similar and the result is

$$x_F[n] = X[0] + \sum_{k=1}^{(N_F-1)/2} \left[X_c[k]\cos(2\pi kn/N_F) + X_s[k]\sin(2\pi kn/N_F) \right] . \quad \text{(G.3)}$$

Equations (G.2) and (G.3) are each a representation of the real-valued signal $x[n]$ in terms of a linear combination of a real constant and real-valued cosines and sines. These are known as the trigonometric forms of the CTFS harmonic function for real-valued signals. The relationships between the complex and trigonometric harmonic functions are

$$X_c[0] = X[0] \ , \ X_s[0] = 0$$
$$X_c[k] = \begin{cases} X[k] + X^*[k] & , \ 0 < k < N_F/2 \\ X[k] & , \ k = N_F/2 \end{cases} \quad \text{(G.4)}$$
$$X_s[k] = \begin{cases} j(X[k] - X^*[k]) & , \ 0 < k < N_F/2 \\ 0 & , \ k = N_F/2 \end{cases}$$

and

$$\mathrm{X}[0] = \mathrm{X}_c[0]$$
$$\mathrm{X}[k] = \begin{cases} \dfrac{\mathrm{X}_c[k] - j\,\mathrm{X}_s[k]}{2} & , \; 0 < k < N_F/2 \\ \mathrm{X}_c[k] & , \; k = N_F/2 \end{cases} \qquad \text{(G.5)}$$
$$\mathrm{X}[-k] = \mathrm{X}^*[k] \; , \; 0 < k \le N_F/2$$

The complex and trigonometric forms of the DTFS are closely related because of Euler's identity $e^{jx} = \cos(x) + j\sin(x)$ which indicates that when we find a complex sinusoid in a DTFS representation of a signal we are, by implication, simultaneously finding a cosine and a sine.

G.2 Properties

Linearity

Let $\mathrm{z}[n] = \alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n]$. Then

$$\mathrm{z}[n] = \alpha \sum_{k=\langle N_0 \rangle} \mathrm{X}[k] e^{j2\pi k F_0 n} + \beta \sum_{k=\langle N_0 \rangle} \mathrm{Y}[k] e^{j2\pi k F_0 n} = \sum_{k=\langle N_0 \rangle} \left(\alpha\,\mathrm{X}[k] + \beta\,\mathrm{Y}[k]\right) e^{j2\pi k F_0 n}$$

But $\mathrm{z}[n]$ also has a DTFS representation,

$$\mathrm{z}[n] = \sum_{k=\langle N_0 \rangle} \mathrm{Z}[k] e^{j2\pi k F_0 n} \quad .$$

Therefore, we can conclude that

$$\mathrm{Z}[k] = \alpha\,\mathrm{X}[k] + \beta\,\mathrm{Y}[k]$$

and

$$\alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n] \xleftrightarrow{\mathcal{FS}} \alpha\,\mathrm{X}[k] + \beta\,\mathrm{Y}[k].$$

Time Shifting

Let $\mathrm{z}[n] = \mathrm{x}[n - n_0]$. Then

$$\mathrm{Z}[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} \mathrm{z}[n] e^{-j2\pi k F_0 n} = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} \mathrm{x}[n - n_0] e^{-j2\pi k F_0 n}$$

Now let $q = n - n_0$ in the x summation. Then, since n covers a range of N_0, q does also and

$$\mathrm{Z}[k] = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} \mathrm{z}[n] e^{-j2\pi kF_0 n} = \frac{1}{N_0}\sum_{q=\langle N_0\rangle} \mathrm{x}[q] e^{-j2\pi kF_0 (q+n_0)}$$

$$\mathrm{Z}[k] = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} \mathrm{z}[n] e^{-j2\pi kF_0 n} = e^{-j2\pi(kF_0)n_0} \underbrace{\frac{1}{N_0}\sum_{q=\langle N_0\rangle} \mathrm{x}[q] e^{-j2\pi kF_0 q}}_{=\mathrm{X}[k]}$$

$$\mathrm{Z}[k] = e^{-j2\pi kF_0 n_0}\,\mathrm{X}[k]$$

and

$$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{FS}} e^{-j2\pi kF_0 n_0}\,\mathrm{X}[k]$$

Frequency Shifting

Let $\mathrm{z}[n] = e^{j2\pi k_0 F_0 n}\,\mathrm{x}[n]$, k_0 an integer. Then

$$\mathrm{Z}[k] = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} \mathrm{z}[n] e^{-j2\pi kF_0 n} = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} e^{j2\pi k_0 F_0 n}\,\mathrm{x}[n] e^{-j2\pi kF_0 n}$$

$$\mathrm{Z}[k] = \underbrace{\frac{1}{N_0}\sum_{n=\langle N_0\rangle} \mathrm{x}[n] e^{-j2\pi(k-k_0)F_0 n}}_{=\mathrm{X}[k-k_0]}$$

$$\mathrm{Z}[k] = \mathrm{X}[k-k_0]$$

$$e^{j2\pi k_0 F_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k-k_0] \quad .$$

Conjugation

Let $\mathrm{z}[n] = \mathrm{x}^*[n]$. Then

$$\mathrm{Z}[k] = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} \mathrm{z}[n] e^{-j2\pi kF_0 n} = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} \mathrm{x}^*[n] e^{-j2\pi kF_0 n}$$

Conjugating both sides,

$$\mathrm{Z}^*[k]=\frac{1}{N_0}\sum_{n=\langle N_0\rangle}\mathrm{x}[n]e^{j2\pi kF_0 n}=\underbrace{\frac{1}{N_0}\sum_{n=\langle N_0\rangle}\mathrm{x}[n]e^{-j2\pi(-kF_0 n)}}_{=\mathrm{X}[-k]}$$

$$\mathrm{Z}^*[k]=\mathrm{X}[-k]$$

or

$$\mathrm{Z}[k]=\mathrm{X}^*[-k]$$

and

$$\mathrm{x}^*[n]\xleftrightarrow{\mathcal{FS}}\mathrm{X}^*[-k]$$

Time Reversal

Let $\mathrm{z}[n]=\mathrm{x}[-n]$. Then

$$\mathrm{Z}[k]=\frac{1}{N_0}\sum_{n=\langle N_0\rangle}\mathrm{z}[n]e^{-j2\pi kF_0 n}=\frac{1}{N_0}\sum_{n=\langle N_0\rangle}\mathrm{x}[-n]e^{-j2\pi kF_0 n}$$

Let $m=-n$. Then if n covers a range of N_0, so does m and

$$\mathrm{Z}[k]=\frac{1}{N_0}\sum_{m=\langle N_0\rangle}\mathrm{x}[-m]e^{-j2\pi kF_0(-m)}=\underbrace{\frac{1}{N_0}\sum_{m=\langle N_0\rangle}\mathrm{x}[-m]e^{-j2\pi(-kF_0)m}}_{=\mathrm{X}[-k]}$$

$$\mathrm{Z}[k]=\mathrm{X}[-k]$$

and

$$\mathrm{x}[-n]\xleftrightarrow{\mathcal{FS}}\mathrm{X}[-k]$$

Time Scaling

Let $\mathrm{z}[n]=\mathrm{x}[an]$, $a>0$. If a is not an integer then some values of $\mathrm{z}[n]$ will be undefined and a DTFS cannot be found for it. If a is an integer, then $\mathrm{z}[n]$ is a decimated version of $\mathrm{x}[n]$ and some of the values of $\mathrm{x}[n]$ do not appear in $\mathrm{z}[n]$. In that case, there cannot be a unique relationship between the harmonic functions of $\mathrm{x}[n]$ and $\mathrm{z}[n]$ through the transformation, $n\to an$ (Figure G-1).

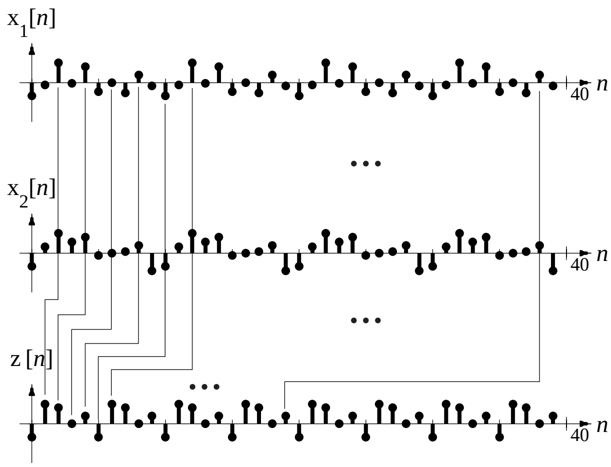

Figure G-1 Two different signals decimated to yield the same signal

However there is an operation for which the relationship between $\mathrm{x}[n]$ and $\mathrm{z}[n]$ is unique. Let m be a positive integer and let

$$\mathrm{z}[n] = \begin{cases} \mathrm{x}[n/m] & , \ n/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}.$$

That is, $\mathrm{z}[n]$ is a time-expanded version of $\mathrm{x}[n]$ formed by placing $m-1$ zeros between adjacent values of $\mathrm{x}[n]$ (Figure G-2).

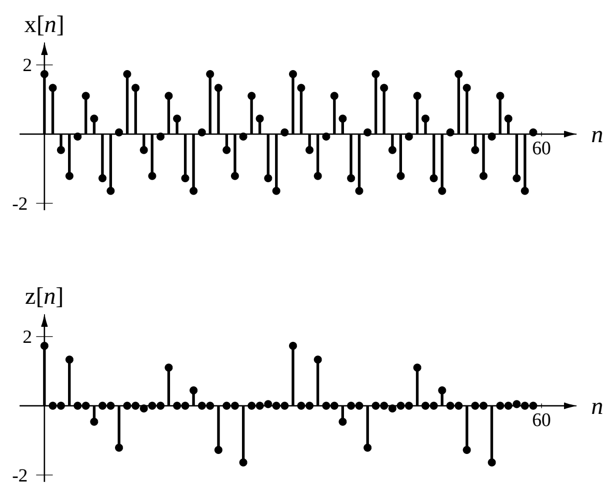

Figure G-2 A DT function and an expanded version formed by inserting zeros between values

If the fundamental period of $\mathrm{x}[n]$ is N_{0x}, the fundamental period of $\mathrm{z}[n]$ is $N_{0z} = mN_{0x}$. Then the DTFS harmonic function for $\mathrm{z}[n]$ with a representation time of $N_F = qN_{0z}$, where q is an integer is

$$\mathrm{Z}[k] = \frac{1}{N_F} \sum_{n=\langle N_F \rangle} \mathrm{z}[n] e^{-j2\pi nk/N_F} .$$

Since all the values of z are zero when n/m is not an integer,

$$\mathrm{Z}[k] = \frac{1}{N_F} \sum_{\substack{n=\langle N_F \rangle \\ n/m \text{ an integer}}} \mathrm{z}[n] e^{-j2\pi nk/N_F} .$$

Let $p = n/m$, when n/m is an integer. Then

$$\mathrm{Z}[k] = \frac{1}{N_F} \sum_{p=\langle N_F/m \rangle} \mathrm{z}[mp] e^{-j2\pi kmp/N_F}$$

and $\mathrm{z}[mp] = \mathrm{x}[p]$. Therefore, since $N_F/m = qN_{0z}/m = qmN_{0x}/m = qN_{0x}$,

$$\mathrm{Z}[k] = \frac{1}{N_F} \sum_{p=\langle qN_{0x} \rangle} \mathrm{x}[p] e^{-j2\pi kp/qN_{0x}} = \frac{1}{m}\mathrm{X}[k]$$

where $\mathrm{X}[k]$ is the harmonic function for $\mathrm{x}[n]$ using a representation time of qN_{0x}. So the time-scaling property is

$$\mathrm{z}[n] = \begin{cases} \mathrm{x}[n/m] & , \; n/m \text{ an integer} \\ 0 & , \; \text{otherwise} \end{cases} .$$
$$N_F \to mN_F \quad , \quad \mathrm{Z}[k] = (1/m)\mathrm{X}[k]$$

Change of Period

If we know that the DTFS harmonic function of $\mathrm{x}[n]$ over the representation time $N_F = mN_{0x}$, where m is an integer, is $\mathrm{X}[k]$ we can find the harmonic function of $\mathrm{x}[n]$ over the representation time qN_F , which is $\mathrm{X}_q[k]$, with q being a positive integer. It is

$$X_q[k] = \frac{1}{qN_F} \sum_{n=\langle qN_F \rangle} x[n] e^{-j2\pi nk/qN_F} .$$

The DT function $x[n]$ has a period N_F and therefore is represented by DT sinusoids at integer multiples of $1/N_F$. The DT function $e^{-j2\pi nk/qN_F}$ has a fundamental period qN_F and fundamental frequency $1/qN_F$. Therefore, on the DT interval $n_0 \le n < n_0 + qN_F$ the two DT functions $x[n]$ and $e^{-j2\pi nk/qN_F}$ are orthogonal unless k/q is an integer. Therefore, for k/q not an integer, $X_q[k] = 0$. For k/q an integer, the summation over qN_F is equivalent to q summations over N_F and

$$X_q[k] = q\left(\frac{1}{qN_F} \sum_{n=\langle N_F \rangle} x[n] e^{-j2\pi nk/qN_F}\right) = \frac{1}{N_F} \sum_{n=\langle N_F \rangle} x[n] e^{-j2\pi nk/qN_F} = X[k/q].$$

Summarizing,

$$N_F \to qN_F, \; q \text{ a positive integer}$$

$$X_q[k] = \begin{cases} X[k/q] & , \; k/q \text{ an integer} \\ 0 & , \; \text{otherwise} \end{cases}$$

(Figure G-3).

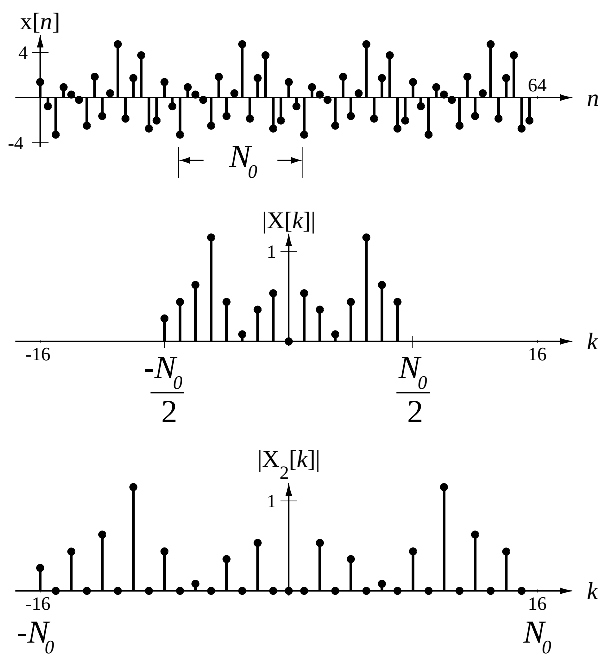

Figure G-3 DT signal and the magnitude of its DTFS harmonic function with $N_F = N_0$ and with $N_F = 2N_0$

Multiplication-Convolution Duality

Let $z[n] = x[n]y[n]$ and let $N_F = mN_{x0} = qN_{y0}$, where m and q are integers. Then

$$Z[k] = \frac{1}{N_F} \sum_{n=\langle N_F \rangle} z[n] e^{-j2\pi kn/N_F} = \frac{1}{N_F} \sum_{n=\langle N_F \rangle} x[n] y[n] e^{-j2\pi kn/N_F} ,$$

and, using

$$y[n] = \sum_{p=\langle N_F \rangle} Y[p] e^{j2\pi pn/N_F} ,$$

$$Z[k] = \frac{1}{N_F} \sum_{n=\langle N_F \rangle} x[n] \sum_{p=\langle N_F \rangle} Y[p] e^{j2\pi pn/N_F} e^{-j2\pi kn/N_F}$$

$$Z[k] = \frac{1}{N_F} \sum_{n=\langle N_F \rangle} x[n] \sum_{p=\langle N_F \rangle} Y[p] e^{-j2\pi (k-p)n/N_F}$$

$$Z[k] = \sum_{p=\langle N_F \rangle} Y[p] \underbrace{\frac{1}{N_F} \sum_{n=\langle N_F \rangle} x[n] e^{-j2\pi (k-p)n/N_F}}_{=X[k-p]}$$

$$Z[k] = \sum_{p=\langle N_F \rangle} Y[p] X[k-p]$$

This result looks just like a convolution sum except that q extends over a finite range instead of an infinite one. This is a *periodic convolution sum* which is indicated by the notation

$$Z[k] = Y[k] \circledast X[k] .$$

Therefore

$$x[n]y[n] \xleftrightarrow[N_F = mN_{x0} = qN_{y0}]{FS} Y[k] \circledast X[k] = \sum_{p=\langle N_F \rangle} Y[p] X[k-p] . \quad \text{(G.6)}$$

$N_{x0} =$

Multiplication of two DT signals corresponds to the convolution of their DTFS harmonic functions but the convolution is now a periodic convolution.

Now let $\mathrm{Z}[k]=\mathrm{Y}[k]\mathrm{X}[k]$ and let $N_F = mN_{x0} = qN_{y0}$, where m and q are integers. Then

$$\mathrm{z}[n]=\sum_{k=\langle N_F\rangle}\mathrm{X}[k]\mathrm{Y}[k]e^{j2\pi kn/N_F}$$

$$\mathrm{z}[n]=\sum_{k=\langle N_F\rangle}\frac{1}{N_F}\sum_{p=\langle N_F\rangle}\mathrm{x}[p]e^{-j2\pi kp/N_F}\,\mathrm{Y}[k]e^{j2\pi kn/N_F}$$

$$\mathrm{z}[n]=\frac{1}{N_F}\sum_{p=\langle N_F\rangle}\mathrm{x}[p]\underbrace{\sum_{k=\langle N_F\rangle}\mathrm{Y}[k]e^{j2\pi k(n-p)/N_F}}_{=\mathrm{y}[n-p]}$$

$$\mathrm{z}[n]=\frac{1}{N_F}\sum_{p=\langle N_F\rangle}\mathrm{x}[p]\mathrm{y}[n-p]$$

or

$$N_F = mN_{x0} = qN_{y0}$$
$$\mathrm{x}[n]\circledast\mathrm{y}[n]\xleftrightarrow{\mathcal{FS}}N_F\,\mathrm{Y}[k]\mathrm{X}[k]\,.$$

Multiplication in either domain corresponds to a periodic convolution sum in the other domain (except for a scale factor of N_F in the case of discrete-time periodic convolution).

First Backward Difference

Let $\mathrm{z}[n]=\mathrm{x}[n]-\mathrm{x}[n-1]$ and let $N_F = mN_{x0} = mN_{z0}$, where m is an integer. Then using the time-shifting property,

$$\mathrm{x}[n-1]\xleftrightarrow{\mathcal{FS}}\mathrm{X}[k]e^{-j2\pi k/N_0}$$

and invoking the linearity property,

$$\mathrm{x}[n]+\mathrm{x}[n-1]\xleftrightarrow{\mathcal{FS}}\mathrm{X}[k]+\mathrm{X}[k]e^{-j2\pi k/N_0}$$

or

$$N_F = mN_0$$
$$\mathrm{x}[n]-\mathrm{x}[n-1]\xleftrightarrow{\mathcal{FS}}\left(1-e^{-j2\pi k/N_0}\right)\mathrm{X}[k]\,.$$

Accumulation

Let $\mathrm{z}[n]=\sum_{m=-\infty}^{n}\mathrm{x}[m]$. It is important for this property to consider the effect of the average value of $\mathrm{x}[n]$. We can write the signal $\mathrm{x}[n]$ as

$$\mathrm{x}[n]=\mathrm{x}_0[n]+\mathrm{X}[0]$$

where $\mathrm{x}_0[n]$ is a signal with an average value of zero and $\mathrm{X}[0]$ is the average value of $\mathrm{x}[n]$. Then

$$\mathrm{z}[n]=\sum_{m=-\infty}^{n}\mathrm{x}_0[m]+\sum_{m=-\infty}^{n}\mathrm{X}[0].$$

Since $\mathrm{X}[0]$ is a constant, $\sum_{m=-\infty}^{n}\mathrm{X}[0]$ increases or decreases linearly with *n*, unless $\mathrm{X}[0]=0$. Therefore, if $\mathrm{X}[0]\neq 0$, $\mathrm{z}[n]$ is not periodic and we cannot find its DTFS. If the average value of $\mathrm{x}[n]$ is zero, $\mathrm{z}[n]$ is periodic and we can find a DTFS for it. Since accumulation is the inverse of the first backward difference,

$$\text{if} \qquad \mathrm{z}[n]=\sum_{m=-\infty}^{n}\mathrm{x}[m] \qquad \text{then} \qquad \mathrm{x}[n]=\mathrm{z}[n]-\mathrm{z}[n-1].$$

But remember, multiple signals can have the same backward difference. For example we just showed that $\mathrm{x}[n]=\mathrm{z}[n]-\mathrm{z}[n-1]$ where $\mathrm{z}[n]=\sum_{m=-\infty}^{n}\mathrm{x}[m]$. But if we redefine $\mathrm{z}[n]$ as $C+\sum_{m=-\infty}^{n}\mathrm{x}[m]$ where C is any constant we can still say that $\mathrm{x}[n]=\mathrm{z}[n]-\mathrm{z}[n-1]$. So, in finding the DTFS of the accumulation of a signal we can find it exactly <u>except for the effect of the constant</u>. The constant only affects $\mathrm{Z}[0]$. So we can relate the harmonic functions of $\mathrm{x}[n]$ and $\mathrm{z}[n]$ except for the $k=0$ values. The first backward difference property proved that $\mathrm{X}[k]=\left(1-e^{-j2\pi k/N_F}\right)\mathrm{Z}[k]$. If follows that

$$\mathrm{Z}[k]=\frac{\mathrm{X}[k]}{1-e^{-j2\pi k/N_F}} \ , \ k\neq 0 \ , \ \text{if } \mathrm{X}[0]=0$$

and

$$N_F = mN_0$$

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \xleftrightarrow{\mathcal{FS}} \frac{\mathrm{X}[k]}{1-e^{-j2\pi k/N_F}} \;,\; k \neq 0 \;,\; \text{if } \mathrm{X}[0]=0$$

Parseval's Theorem

The total signal energy of a periodic signal $\mathrm{x}[n]$ is infinite (unless it is the trivial signal $\mathrm{x}[n]=0$). The signal energy over one period $N_F = mN_{x0}$ is defined as

$$E_{x,N_F} = \sum_{n=\langle N_F \rangle} \left|\mathrm{x}[n]\right|^2 = \sum_{n=\langle N_F \rangle} \left| \sum_{k=\langle N_F \rangle} \mathrm{X}[k] e^{j2\pi kn/N_F} \right|^2$$

$$E_{x,N_F} = \sum_{n=n_0}^{n_0+N_F-1} \left(\sum_{k=\langle N_F \rangle} \mathrm{X}[k] e^{j2\pi kn/N_F} \right) \left(\sum_{q=\langle N_F \rangle} \mathrm{X}[q] e^{j2\pi qn/N_F} \right)^*$$

$$E_{x,N_F} = \sum_{n=\langle N_F \rangle} \left(\sum_{k=\langle N_F \rangle} \left|\mathrm{X}[k]\right|^2 + \sum_{\substack{k=\langle N_F \rangle \\ k \neq q}} \sum_{q=\langle N_F \rangle} \mathrm{X}[k] e^{j2\pi kn/N_F} \mathrm{X}^*[q] e^{-j2\pi qn/N_F} \right)$$

$$E_{x,N_F} = \sum_{n=\langle N_F \rangle} \left(\sum_{k=\langle N_F \rangle} \left|\mathrm{X}[k]\right|^2 + \underbrace{\sum_{\substack{k=\langle N_F \rangle \\ k \neq q}} \sum_{q=\langle N_F \rangle} \mathrm{X}[k] \mathrm{X}^*[q] e^{j2\pi (k-q) n/N_F}}_{=0} \right)$$

$$E_{x,N_F} = N_F \sum_{k=\langle N_F \rangle} \left|\mathrm{X}[k]\right|^2$$

Then

$$N_F = mN_0$$

$$\frac{1}{N_F} \sum_{n=\langle N_F \rangle} \left|\mathrm{x}[n]\right|^2 = \sum_{k=\langle N_F \rangle} \left|\mathrm{X}[k]\right|^2$$

which, in words, says that the average signal power of the signal is equal to the sum of the average signal powers in its DTFS harmonics.