Web Appendix H - Derivations of the Properties of the Continuous-Time Fourier Transform

H.1 Numerical Computation of the CTFT

In cases in which the signal to be transformed is not readily describable by a mathematical function or the Fourier-transform integral cannot be done analytically, we can sometimes find an approximation to the CTFT numerically using the DFT which was first introduced in Chapter 8. The CTFT of a signal $\mathrm{x}(t)$ is

$$\mathrm{X}(f) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j2\pi ft} dt \qquad .$$

If we apply this to a signal that is causal we get

$$\mathrm{X}(f) = \int_{0}^{\infty} \mathrm{x}(t) e^{-j2\pi ft} dt .$$

We can write this integral in the form

$$\mathrm{X}(f) = \sum_{n=0}^{\infty} \int_{nT_s}^{(n+1)T_s} \mathrm{x}(t) e^{-j2\pi ft} dt .$$

If T_s is small enough, the variation of $\mathrm{x}(t)$ in the time interval $nT_s \le t < (n+1)T_s$ is small and the CTFT can be approximated by

$$\mathrm{X}(f) \cong \sum_{n=0}^{\infty} \mathrm{x}(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-j2\pi ft} dt .$$

or

$$\mathrm{X}(f) \cong \sum_{n=0}^{\infty} \mathrm{x}(nT_s) \frac{e^{-j2\pi fnT_s} - e^{-j2\pi f(n+1)T_s}}{j2\pi f}$$

or

$$\mathrm{X}(f) \cong \frac{1 - e^{-j2\pi fT_s}}{j2\pi f} \sum_{n=0}^{\infty} \mathrm{x}(nT_s) e^{-j2\pi fnT_s} = T_s e^{-j\pi fT_s} \operatorname{sinc}(T_s f) \sum_{n=0}^{\infty} \mathrm{x}(nT_s) e^{-j2\pi fnT_s}$$
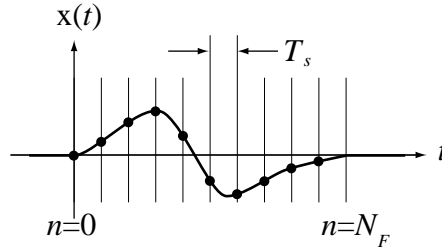
(Figure H-1).

Figure H-1 A signal and multiple intervals on which the CTFT integral can be evaluated

If $x(t)$ is an energy signal then beyond some finite time its size must become negligible and we can replace the infinite range of n in the summation with a finite range $0 \le n < N_F$ yielding

$$X(f) \cong T_s e^{-j\pi f T_s} \operatorname{sinc}(T_s f) \sum_{n=0}^{N_F - 1} x(nT_s) e^{-j2\pi f n T_s} .$$

Now if we compute the CTFT only at integer multiples of $f_s / N_F = f_F$, which is the frequency-domain resolution of this approximation to the CTFT,

$$X(kf_F) \cong T_s e^{-j\pi k f_F T_s} \operatorname{sinc}(T_s k f_F) \sum_{n=0}^{N_F - 1} x(nT_s) e^{-j2\pi k n f_F T_s}$$

or

$$X(kf_F) \cong T_s e^{-j\pi k / N_F} \operatorname{sinc}(k / N_F) \sum_{n=0}^{N_F - 1} x(nT_s) e^{-j2\pi k n / N_F} .$$

The summation in this equation is the DFT of $x(nT_s)$. Therefore

$$X(kf_F) \cong T_s e^{-j\pi k / N_F} \operatorname{sinc}(k / N_F) \times \mathsf{DFT}\left(x(nT_s)\right)$$

where the notation $\mathsf{DFT}(\cdot)$ means "discrete Fourier transform of". For $|k| << N_F$,

$$X(kf_F) \cong T_s \times \mathsf{DFT}\left(x(nT_s)\right) . \tag{H.1}$$

So if the signal to be transformed is a causal energy signal and we sample it over a time containing practically all of its energy and if the samples are close enough together that the signal does not change appreciably between samples, the approximation in (H.1) becomes accurate for $|k| << N_F$.

H.2 Linearity

Let α and β be constants and let $z(t) = \alpha x(t) + \beta y(t)$. Then

$$Z(f) = \int_{-\infty}^{\infty}\left[\alpha x(t) + \beta y(t)\right]e^{-j2\pi ft}dt = \alpha\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt + \beta\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt = \alpha X(f) + \beta y(f)$$

and the linearity property is

$$\alpha x(t) + \beta y(t)\xleftrightarrow{\ F\ }\alpha X(f) + \beta y(f)$$

or

$$\alpha x(t) + \beta y(t)\xleftrightarrow{\ F\ }\alpha X(j\omega) + \beta y(j\omega).$$

H.3 Time Shifting and Frequency Shifting

Let t_0 be any real constant and let $z(t) = x(t - t_0)$. Then the CTFT of $z(t)$ is

$$Z(f) = \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(t - t_0)e^{-j2\pi ft}dt$$

Make the change of variable $\tau = t - t_0$ and $d\tau = dt$. Then

$$Z(f) = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f(\tau + t_0)}d\tau = e^{-j2\pi ft_0}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau = e^{-j2\pi ft_0} X(f)$$

and the time-shifting property is

$$x(t - t_0)\xleftrightarrow{\ F\ }X(f)e^{-j2\pi ft_0}$$

or

$$x(t - t_0)\xleftrightarrow{\ F\ }X(j\omega)e^{-j\omega t_0}\ .$$

Let f_0 be any real constant and let $Z(f) = X(f - f_0)$. Then $z(t)$ is

$$z(t) = \int_{-\infty}^{\infty} Z(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f - f_0)e^{j2\pi ft}df$$

Make the change of variable $\phi = f - f_0$ and $d\phi = df$. Then

$$z(t) = \int_{-\infty}^{\infty} X(\phi)e^{j2\pi(\phi + f_0)t}d\phi = e^{j2\pi f_0 t}\int_{-\infty}^{\infty} X(\phi)e^{j2\pi\phi t}d\phi = e^{j2\pi f_0 t} x(t)$$

and the time-shifting property is

$$x(t)e^{+j2\pi f_0 t} \xleftrightarrow{\quad F \quad} X(f - f_0)$$

or

$$x(t)e^{+j\omega_0 t} \xleftrightarrow{\quad F \quad} X(j(\omega - \omega_0)) \ .$$

H.4 Time Scaling and Frequency Scaling

Let a be any real constant other than zero and let $z(t) = x(at)$. Then the CTFT of $z(t)$ is

$$Z(f) = \int_{-\infty}^{\infty} z(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(at)e^{-j2\pi ft}dt$$

Make the change of variable $\tau = at$ and $d\tau = a\,dt$. Then, if $a > 0$,

$$Z(f) = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau/a}\frac{d\tau}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi(f/a)\tau}d\tau = \frac{1}{a}X\left(\frac{f}{a}\right)$$

and if $a < 0$,

$$Z(f) = \int_{\infty}^{-\infty} x(\tau)e^{-j2\pi f\tau/a}\frac{d\tau}{a} = -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi(f/a)\tau}d\tau = -\frac{1}{a}X\left(\frac{f}{a}\right).$$

Therefore, in either case, $Z(f) = (1/|a|)X(f/a)$ and the time-scaling property is

$$x(at) \xleftrightarrow{\quad F \quad} (1/|a|)X(f/a) \quad \text{or} \quad x(at) \xleftrightarrow{\quad F \quad} (1/|a|)X(j\omega/a)$$

In the time-scaling property, let $b = 1/a$. Then

$$x(t/b) \xleftrightarrow{\quad F \quad} |b|X(bf) \quad \text{or} \quad x(t/b) \xleftrightarrow{\quad F \quad} |b|X(jb\omega)$$

and the frequency scaling property is

$$(1/|b|)x(t/b) \xleftrightarrow{\quad F \quad} X(bf) \quad \text{or} \quad (1/|b|)x(t/b) \xleftrightarrow{\quad F \quad} X(jb\omega) \ .$$

H.5 Transform of a Conjugate

The inverse CTFT of $X(f)$ is,

$$\mathcal{F}^{-1}\big(X(f)\big) = x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \ .$$

The complex conjugate of $x(t)$ is

$$x^*(t) = \left[\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df\right]^* = \int_{-\infty}^{\infty} X^*(f) e^{-j2\pi ft} df \ .$$

Make the change of variable $\phi = -f$ and $d\phi = -df$ we get

$$x^*(t) = -\int_{\infty}^{-\infty} X^*(-\phi) e^{j2\pi \phi t} d\phi = \underbrace{\int_{-\infty}^{\infty} X^*(-\phi) e^{j2\pi \phi t} d\phi}_{\mathcal{F}^{-1}\left[X^*(-f)\right]} \ .$$

The conjugation property is

$$x^*(t) \xleftrightarrow{\ \mathcal{F}\ } X^*(-f) \quad \text{or} \quad x^*(t) \xleftrightarrow{\ \mathcal{F}\ } X^*(-j\omega) \ .$$

H.6 Multiplication - Convolution Duality

Let the convolution of $x(t)$ and $y(t)$ be

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau \ .$$

The CTFT of $z(t)$ is

$$Z(f) = \int_{-\infty}^{\infty} z(t) e^{-j2\pi ft} dt$$

or

$$Z(f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau\right] e^{-j2\pi ft} dt \ .$$

Reversing the order of integration,

$$Z(f) = \int_{-\infty}^{\infty} x(\tau) \underbrace{\left[\int_{-\infty}^{\infty} y(t - \tau) e^{-j2\pi ft} dt\right]}_{\mathcal{F}\left[y(t-\tau)\right]} d\tau \ .$$

Then, using the time-shifting property,

$$Z(f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} Y(f) d\tau$$

Since $Y(f)$ is not a function of τ,

$$Z(f) = Y(f) \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau}_{F[x(\tau)]},$$

and finally $Z(f) = X(f) Y(f)$. The time-domain convolution property is

$$x(t) * y(t) \overset{F}{\longleftrightarrow} X(f) Y(f) \quad \text{or} \quad x(t) * y(t) \overset{F}{\longleftrightarrow} X(j\omega) Y(j\omega)$$

Let the convolution of $X(f)$ and $Y(f)$ be

$$Z(f) = X(f) * Y(f).$$

Then $z(t)$ is

$$z(t) = \int_{-\infty}^{\infty} Z(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} X(f) * Y(f) e^{j2\pi ft} df$$

or

$$z(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} X(\phi) Y(f - \phi) d\phi \right] e^{j2\pi ft} df.$$

Reversing the order of integration,

$$z(t) = \int_{-\infty}^{\infty} X(\phi) \underbrace{\left[\int_{-\infty}^{\infty} Y(f - \phi) e^{j2\pi ft} df \right]}_{e^{j2\pi \phi t} y(t)} d\phi.$$

Then, using the frequency-shifting property,

$$z(t) = y(t) \underbrace{\int_{-\infty}^{\infty} X(\phi) e^{j2\pi \phi t} d\phi}_{= x(t)} = x(t) y(t)$$

and the time-domain multiplication property is

$$x(t) y(t) \overset{F}{\longleftrightarrow} X(f) * Y(f) \quad \text{or} \quad x(t) y(t) \overset{F}{\longleftrightarrow} \frac{1}{2\pi} X(j\omega) * Y(j\omega).$$

The extra $1/2\pi$ is there because of the change of variable in converting f to ω.

H.7 Time Differentiation

The time-domain function $\mathrm{x}(t)$ can be expressed as

$$\mathrm{x}(t) = \int_{-\infty}^{\infty} \mathrm{X}(f) e^{j2\pi ft} df \ .$$

Differentiating both sides with respect to time,

$$\frac{d}{dt}\left(\mathrm{x}(t)\right) = \frac{d}{dt} \int_{-\infty}^{\infty} \mathrm{X}(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} j2\pi f \, \mathrm{X}(f) e^{j2\pi ft} df = \mathsf{F}^{-1}\left(j2\pi f \, \mathrm{X}(f)\right)$$

Therefore the time-differentiation property is

$$\frac{d}{dt}\left(\mathrm{x}(t)\right) \xleftrightarrow{\ \mathsf{F}\ } j2\pi f \, \mathrm{X}(f) \quad \text{or} \quad \frac{d}{dt}\left(\mathrm{x}(t)\right) \xleftrightarrow{\ \mathsf{F}\ } j\omega \, \mathrm{X}(j\omega) \ .$$

H.8 Transforms of Periodic Signals

If a time signal $\mathrm{x}(t)$ is periodic it can be represented exactly by a complex CTFS (by letting $T_F = T_0$). Therefore, for periodic signals (using the frequency-shifting property),

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{-j2\pi(kf_F)t} \xleftrightarrow{\ \mathsf{F}\ } \mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta\left(f - kf_F\right) \ ,$$

or

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{-j(k\omega_F)t} \xleftrightarrow{\ \mathsf{F}\ } \mathrm{X}(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta\left(\omega - k\omega_F\right) \ .$$

H.9 Parseval's Theorem

The product of two energy signals in the time domain corresponds to the convolution of their transforms in the frequency domain.

$$\mathsf{F}\left[\mathrm{x}(t)\mathrm{y}(t)\right] = \mathrm{X}(f) * \mathrm{Y}(f) = \int_{-\infty}^{\infty} \mathrm{X}(\phi)\mathrm{Y}(f - \phi)\,d\phi \quad . \tag{H.2}$$

From the definition of the CTFT,

$$\mathsf{F}\left[\mathrm{x}(t)\mathrm{y}(t)\right] = \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}(t)e^{-j2\pi ft}\,dt \quad . \tag{H.3}$$

Combining (H.2) and (H.3),

$$\int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}(t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} X(\phi)\mathrm{Y}(f - \phi)\,d\phi .$$

This relation holds for any value of f. Setting $f = 0$,

$$\int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}(t)\,dt = \int_{-\infty}^{\infty} \mathrm{X}(\phi)\mathrm{Y}(-\phi)\,d\phi .$$

This is known as the generalized form of Parseval's theorem. For the special case in which $\mathrm{y}(t) = \mathrm{x}^*(t)$, this becomes

$$\int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{x}^*(t)\,dt = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2\,dt = \int_{-\infty}^{\infty} \mathrm{X}(\phi)\mathrm{X}(-\phi)\,d\phi = \int_{-\infty}^{\infty} \mathrm{X}(\phi)\mathrm{X}^*(\phi)\,d\phi = \int_{-\infty}^{\infty} |\mathrm{X}(\phi)|^2\,d\phi$$

where we have used $\mathrm{y}\mathrm{y}^* = |\mathrm{y}|^2$ and $\mathrm{y}^*(t) \xleftrightarrow{\ \mathsf{F}\ } \mathrm{Y}^*(-f)$. So, ultimately, we have the equality of total signal energy in the time and frequency domains

$$\int_{-\infty}^{\infty} |\mathrm{x}(t)|^2\,dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2\,df \quad .$$

H.10 Integral Definition of an Impulse

The definition of the Fourier transform pair

$$\mathrm{X}(f) = \mathsf{F}\left(\mathrm{x}(t)\right) = \int_{-\infty}^{\infty} \mathrm{x}(t)e^{-j2\pi ft}\,dt \quad , \quad \mathrm{x}(t) = \mathsf{F}^{-1}\left(\mathrm{X}(f)\right) = \int_{-\infty}^{\infty} \mathrm{X}(f)e^{+j2\pi ft}\,df$$

can be used to prove a handy result. Since

$$x(t) = \int\limits_{-\infty}^{\infty} X(f) e^{+j2\pi ft} df \qquad\qquad (H.4)$$

and (making the change of variable $t \to \tau$ in the definition of the forward CTFT)

$$X(f) = \int\limits_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \qquad\qquad (H.5)$$

we can combine (H.4) and (H.5), to get

$$x(t) = \int\limits_{-\infty}^{\infty} \left[\int\limits_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau \right] e^{+j2\pi ft} df$$

or

$$x(t) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x(\tau) e^{-j2\pi f(t-\tau)} d\tau df \ . \qquad\qquad (H.6)$$

Rearranging to do the f integration first in (H.6),

$$x(t) = \int\limits_{-\infty}^{\infty} x(\tau) \left[\int\limits_{-\infty}^{\infty} e^{-j2\pi f(t-\tau)} df \right] d\tau \quad . \qquad\qquad (H.7)$$

In words, this integral says that if we take any arbitrary signal x(t) replace t by τ, and multiply that by a function $\int_{-\infty}^{\infty} e^{-j2\pi f(t-\tau)} df$ and then integrate over all τ we get x(t) back. We could re-write (H.7) in the form

$$x(t) = \int\limits_{-\infty}^{\infty} x(\tau) g(t-\tau) d\tau$$

where

$$g(t) = \int\limits_{-\infty}^{\infty} e^{-j2\pi ft} df \ .$$

Recall the sampling property of the impulse

$$\int\limits_{-\infty}^{\infty} g(t) \delta(t - t_0) dt = g(t_0) \ .$$

The only way the equality

$$x(t) = \int\limits_{-\infty}^{\infty} x(\tau) g(t-\tau) d\tau$$

can be satisfied is if $g(t - \tau) = \delta(t - \tau)$. That is, if

$$\int_{-\infty}^{\infty} e^{-j2\pi f(t-\tau)} df = \delta(t - \tau) .$$

This is another valid definition of a unit impulse. A more common form of this result is

$$\int_{-\infty}^{\infty} e^{-j2\pi xy} dy = \delta(x) .$$

H.11 Duality

If the CTFT of $x(t)$ is $X(f)$, what is the CTFT of $X(t)$?

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau$$

Therefore, replacing f by t in the right side,

$$X(t) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi t\tau} d\tau .$$

The CTFT is

$$F\left(X(t)\right) = F\left[\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi t\tau} d\tau\right] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi t\tau} d\tau\right] e^{-j2\pi ft} dt$$

or

$$F\left(X(t)\right) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} e^{-j2\pi(\tau + f)t} dt d\tau . \tag{H.8}$$

Using the integral definition of an impulse,

$$\int_{-\infty}^{\infty} e^{-j2\pi xy} dy = \delta(x) ,$$

we can write

$$\int_{-\infty}^{\infty} e^{-j2\pi t(\tau + f)} dt = \delta(\tau + f) .$$

Now, substituting into (H.8)

$$\mathcal{F}\left(X(t)\right) = \int_{-\infty}^{\infty} x(\tau)\delta(\tau + f)d\tau = x(-f).$$

The duality property is

$$X(t)\xleftrightarrow{\ \ F\ \ }x(-f) \quad \text{and} \quad X(-t)\xleftrightarrow{\ \ F\ \ }x(f)$$

or

$$X(jt)\xleftrightarrow{\ \ F\ \ }2\pi x(-\omega) \quad \text{and} \quad X(-jt)\xleftrightarrow{\ \ F\ \ }2\pi x(\omega) .$$

The proof of $X(-t)\xleftrightarrow{\ \ F\ \ }x(f)$ is similar.

H.12 Total-Area Integral Using Fourier Transforms

The total area under a time- or frequency-domain signal can be found by evaluating its CTFT or inverse CTFT with an argument of zero.

$$X(0) = \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt\right]_{f\to 0} = \int_{-\infty}^{\infty} x(t)dt$$

and

$$x(0) = \left[\int_{-\infty}^{\infty} X(f)e^{+j2\pi ft}df\right]_{t\to 0} = \int_{-\infty}^{\infty} X(f)df$$

or

$$X(0) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]_{\omega\to 0} = \int_{-\infty}^{\infty} x(t)dt$$

and

$$x(0) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{+j\omega t}d\omega\right]_{t\to 0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$$

H.13 Integration

Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. We know that

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau .$$

Using multiplication-convolution duality,

$$y(t) = x(t) * u(t) \overset{F}{\longleftrightarrow} Y(j\omega) = X(j\omega)U(j\omega) \qquad \text{(H.9)}$$

Using the Fourier pair

$$u(t) \overset{F}{\longleftrightarrow} 1/j\omega + \pi\delta(\omega)$$

from (H.9)

$$Y(j\omega) = \frac{X(j\omega)}{j\omega} + \pi\delta(\omega)X(j\omega) = \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Making the change of variable, $\omega = 2\pi f$ the cyclic frequency form is

$$Y(f) = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f).$$

Therefore the Fourier pairs are

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\longleftrightarrow} \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

and

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\longleftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega) \ .$$