Web Appendix I - Derivations of the Properties of the Discrete-Time Fourier Transform

I.1 Linearity

Let $\mathrm{z}[n] = \alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n]$ where α and β are constants. Then

$$\mathrm{Z}(F) = \sum_{n=-\infty}^{\infty} \left(\alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n]\right) e^{-j2\pi Fn} = \alpha \sum_{n=-\infty}^{\infty} \mathrm{x}[n] e^{-j2\pi Fn} + \beta \sum_{n=-\infty}^{\infty} \mathrm{y}[n] e^{-j2\pi Fn} = \alpha\,\mathrm{X}(F) + \beta\,\mathrm{Y}(F)$$

and the linearity property is

$$\alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(F) + \beta\,\mathrm{Y}(F) \; .$$

I.2 Time Shifting and Frequency Shifting

Let $\mathrm{z}[n] = \mathrm{x}[n - n_0]$. Then

$$\mathrm{Z}(F) = \sum_{n=-\infty}^{\infty} \mathrm{z}[n] e^{-j2\pi Fn} = \sum_{n=-\infty}^{\infty} \mathrm{x}[n - n_0] e^{-j2\pi Fn} \; .$$

Let $m = n - n_0$. Then

$$\mathrm{Z}(F) = \sum_{n=-\infty}^{\infty} \mathrm{x}[m] e^{-j2\pi F(m+n_0)} = e^{-j2\pi Fn_0} \sum_{n=-\infty}^{\infty} \mathrm{x}[m] e^{-j2\pi Fm} = e^{-j2\pi Fn_0}\,\mathrm{X}(F)$$

and the time shifting property is

$$\mathrm{x}[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j2\pi Fn_0}\,\mathrm{X}(F) \text{ or } \mathrm{x}[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\Omega n_0}\,\mathrm{X}(e^{j\Omega}) \, .$$

Let $\mathrm{Z}(F) = \mathrm{X}(F - F_0)$. Then

$$\mathrm{z}[n] = \int_1 \mathrm{Z}(F) e^{j2\pi Fn} dF = \int_1 \mathrm{X}(F - F_0) e^{j2\pi Fn} dF \; .$$

Let $\Phi = F - F_0$. Then

$$z[n] = \int_1 X(\Phi) e^{j2\pi(\Phi + F_0)n} d\Phi = e^{j2\pi F_0 n} \int_1 X(\Phi) e^{j2\pi\Phi n} d\Phi = e^{j2\pi F_0 n} \, x[n]$$

and the frequency shifting property is

$$e^{j2\pi F_0 n} \, x[n] \xleftrightarrow{\mathcal{F}} X(F - F_0) \text{ or } e^{j2\pi F_0 n} \, x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j(\Omega - \Omega_0)}\right).$$

I.3 Time and Frequency Scaling

Let

$$z[n] = \begin{cases} x[n/m] & , \ n/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

where m is an integer. Then

$$Z(F) = \sum_{n=-\infty}^{\infty} z[n] e^{-j2\pi Fn} = \sum_{\substack{n=-\infty \\ n/m \text{ is an integer}}}^{\infty} x[n/m] e^{-j2\pi Fn}$$

Let $p = n/m$, then $x[p] = x[n/m]$, for every integer value of p and zero for every non-integer value of p and

$$Z(F) = \sum_{p=-\infty}^{\infty} x[p] e^{-j2\pi Fmp} = X(mF)$$

Therefore

$$z[n] \xleftrightarrow{\mathcal{F}} X(mF) \text{ or } z[n] \xleftrightarrow{\mathcal{F}} X\left(e^{jm\Omega}\right)$$

I.4 Transform of a Conjugate

$$\mathcal{F}\left(x^*[n]\right) = \sum_{n=-\infty}^{\infty} x^*[n] e^{-j2\pi Fn} = \left(\sum_{n=-\infty}^{\infty} x[n] e^{+j2\pi Fn}\right)^* = X^*(-F)$$

$$x^*[n] \xleftrightarrow{\mathcal{F}} X^*(-F) \text{ or } x^*[n] \xleftrightarrow{\mathcal{F}} X^*\left(e^{-j\Omega}\right) \quad \text{(I.1)}$$

I.5 Differencing and Accumulation

Using the time-shifting property

$$\mathcal{F}\left(\mathrm{x}[n]-\mathrm{x}[n-1]\right)=\mathrm{X}(F)-e^{-j2\pi F}\,\mathrm{X}(F)=\left(1-e^{-j2\pi F}\right)\mathrm{X}(F)$$

$$\mathrm{x}[n]-\mathrm{x}[n-1]\xleftrightarrow{\mathcal{F}}\left(1-e^{-j2\pi F}\right)\mathrm{X}(F)$$

or

$$\mathrm{x}[n]-\mathrm{x}[n-1]\xleftrightarrow{\mathcal{F}}\left(1-e^{-j\Omega}\right)\mathrm{X}\left(e^{j\Omega}\right).$$

Let $\mathrm{z}[n]=\sum_{m=-\infty}^{n}\mathrm{x}[m]$. Using the fact that $\sum_{m=-\infty}^{n}\mathrm{x}[m]=\mathrm{x}[n]*\mathrm{u}[n]$,

$$\mathrm{z}[n]=\mathrm{x}[n]*\mathrm{u}[n]\Rightarrow \mathrm{Z}(F)=\mathrm{X}(F)\mathrm{U}(F).$$

Using $\mathrm{U}(F)=\dfrac{1}{1-e^{-j2\pi F}}+\dfrac{1}{2}\delta_1(F)$,

$$\mathrm{Z}(F)=\mathrm{X}(F)\left[\frac{1}{1-e^{-j2\pi F}}+\frac{1}{2}\delta_1(F)\right]=\left[\frac{\mathrm{X}(F)}{1-e^{-j2\pi F}}+\frac{1}{2}\mathrm{X}(0)\delta_1(F)\right].$$

and the accumulation property of the DTFT is

$$\sum_{m=-\infty}^{n}\mathrm{x}[m]\xleftrightarrow{\mathcal{F}}\frac{\mathrm{X}(F)}{1-e^{-j2\pi F}}+\frac{1}{2}\mathrm{X}(0)\delta_1(F)$$

or

$$\sum_{m=-\infty}^{n}\mathrm{x}[m]\xleftrightarrow{\mathcal{F}}\frac{\mathrm{X}\left(e^{j\Omega}\right)}{1-e^{-j\Omega}}+\pi\,\mathrm{X}\left(e^{j0}\right)\delta_{2\pi}(\Omega).$$

I.6 Time Reversal

$$\mathcal{F}\left(\mathrm{x}[-n]\right)=\sum_{n=-\infty}^{\infty}\mathrm{x}[-n]e^{-j2\pi Fn}$$

Let $m=-n$. Then

$$\mathcal{F}\left(\mathrm{x}[-n]\right)=\sum_{m=\infty}^{-\infty}\mathrm{x}[m]e^{+j2\pi Fm}=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]e^{-j2\pi(-F)m}=\mathrm{X}(-F)$$

$$\mathrm{x}[-n]\xleftrightarrow{\mathcal{F}}\mathrm{X}(-F)\ \text{ or }\ \mathrm{x}[-n]\xleftrightarrow{\mathcal{F}}\mathrm{X}\left(e^{-j\Omega}\right)$$

I.7 Multiplication - Convolution Duality

Let

$$z[n] = x[n] * y[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m].$$

Then

$$Z(F) = \sum_{n=-\infty}^{\infty} z[n]e^{-j2\pi Fn} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[m]y[n-m]e^{-j2\pi Fn}.$$

Reversing the order of summation,

$$Z(F) = \sum_{m=-\infty}^{\infty} x[m]\underbrace{\sum_{n=-\infty}^{\infty} y[n-m]e^{-j2\pi Fn}}_{\mathcal{F}(y[n-m])} = \sum_{m=-\infty}^{\infty} x[m]Y(F)e^{-j2\pi Fm}$$

$$Z(F) = Y(F)\underbrace{\sum_{m=-\infty}^{\infty} x[m]e^{-j2\pi Fm}}_{\mathcal{F}(x[m])} = Y(F)X(F).$$

Therefore

$$x[n] * y[n] \xleftrightarrow{\mathcal{F}} X(F)Y(F)$$

or

$$x[n] * y[n] \xleftrightarrow{\mathcal{F}} X(j\Omega)Y(j\Omega).$$

Let

$$z[n] = x[n]y[n].$$

Then

$$Z(F) = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-j2\pi Fn}.$$

$$Z(F) = \sum_{n=-\infty}^{\infty}\left(\int_1 X(\Phi)e^{j2\pi\Phi n}d\Phi\right)y[n]e^{-j2\pi Fn} = \int_1 X(\Phi)\sum_{n=-\infty}^{\infty} e^{j2\pi\Phi n}y[n]e^{-j2\pi Fn}d\Phi$$

$$Z(F) = \int_1 X(\Phi)\underbrace{\sum_{n=-\infty}^{\infty} y[n]e^{-j2\pi(F-\Phi)n}}_{Y(F-\lambda)}d\Phi = \int_1 X(\Phi)Y(F-\Phi)d\Phi$$

The last integral $\int_1 X(\Phi)Y(F-\Phi)d\Phi$ is another instance of periodic convolution. Therefore

$$\mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F) \circledast \mathrm{Y}(F) \quad \text{or} \quad \mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathrm{X}(e^{j\Omega}) \circledast \mathrm{Y}(e^{j\Omega}) \tag{I.2}$$

I.8 Accumulation Definition of a Periodic Impulse

The CTFT leads to an integral definition of an impulse. In a similar manner the DTFT leads to an accumulation definition of a periodic impulse. Begin with the definition

$$\mathrm{X}(F) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] e^{-j2\pi Fn} \quad \text{and} \quad \mathrm{x}[n] = \int_1 \mathrm{X}(F) e^{j2\pi Fn} dF \tag{I.3}$$

Then, in

$$\mathrm{X}(F) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] e^{-j2\pi Fn} \tag{I.4}$$

replace $\mathrm{x}[n]$ by its integral equivalent $\mathrm{x}[n] = \int_1 \mathrm{X}(\Phi) e^{j2\pi\Phi n} d\Phi$ (changing F to Φ to avoid confusion between the two F's appearing in the right-hand equation in (I.3) and in (I.4) which have different meaning in the two equations).

$$\mathrm{X}(F) = \sum_{n=-\infty}^{\infty} \left[\int_1 \mathrm{X}(\Phi) e^{j2\pi\Phi n} d\Phi \right] e^{-j2\pi Fn} = \sum_{n=-\infty}^{\infty} \int_{\Phi_0}^{\Phi_0+1} \mathrm{X}(\Phi) e^{j2\pi(\Phi-F)n} d\Phi .$$

or

$$\mathrm{X}(F) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{X}_p(\Phi) e^{j2\pi(\Phi-F)n} d\Phi = \sum_{n=-\infty}^{\infty} \mathrm{X}_p(F) * e^{-j2\pi Fn}$$

or

$$\mathrm{X}(F) = \mathrm{X}_p(F) * \sum_{n=-\infty}^{\infty} e^{-j2\pi Fn} \tag{I.5}$$

where

$$\mathrm{X}_p(F) = \begin{cases} \mathrm{X}(F) & , \ F_0 < F < F_0 + 1 \\ 0 & , \ \text{otherwise} \end{cases}$$

is any arbitrary single period of $\mathrm{X}(F)$. Since $\mathrm{X}_p(F)$ is one period of $\mathrm{X}(F)$ and the period is one, it follows that

$$\mathrm{X}(F) = \mathrm{X}_p(F) * \delta_1(F) . \tag{I.6}$$

Therefore, if (I.5) and (I.6) are both true that means that

$$\sum_{n=-\infty}^{\infty} e^{-j2\pi Fn} = \delta_1(F)$$

and, since $\delta_1(F)$ is an even function,

$$\sum_{n=-\infty}^{\infty} e^{j2\pi Fn} = \delta_1(F).$$

I.9 Parseval's Theorem

The total signal energy in $\mathrm{x}[n]$ is

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \sum_{n=-\infty}^{\infty} \left|\int_1 \mathrm{X}(F) e^{j2\pi Fn} dF\right|^2 = \sum_{n=-\infty}^{\infty} \left(\int_1 \mathrm{X}(F) e^{j2\pi Fn} dF\right)\left(\int_1 \mathrm{X}(\Phi) e^{j2\pi \Phi n} d\Phi\right)^*$$

or

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \sum_{n=-\infty}^{\infty} \int_1 \mathrm{X}(F) \int_1 \mathrm{X}^*(\Phi) e^{-j2\pi(\Phi-F)n} d\Phi dF .$$

We can exchange the order of summation and integration to yield

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \int_1 \mathrm{X}(F) \int_1 \mathrm{X}^*(\Phi) \underbrace{\sum_{n=-\infty}^{\infty} e^{-j2\pi(\Phi-F)n}}_{=\delta_1(\Phi-F)} d\Phi dF$$

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \int_1 \mathrm{X}(F) \int_1 \mathrm{X}^*(\Phi) \delta(\Phi - F) d\Phi dF$$

and

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \int_1 \mathrm{X}(F) \mathrm{X}^*(F) dF = \int_1 \left|\mathrm{X}(F)\right|^2 dF ,$$

proving that the total energy over all discrete-time n is equal to the total energy in one fundamental period of DT frequency F (that fundamental period being one for any DTFT). The equivalent result for the radian-frequency form of the DTFT is

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \frac{1}{2\pi} \int_{2\pi} \left|\mathrm{X}(e^{j\Omega})\right|^2 d\Omega .$$