

Web Appendix J - Frequency Response of an Ideal Lowpass Filter

The ideal lowpass filter makes a transition from allowing signals through at frequencies below its cutoff frequency with no distortion at all and completely blocking frequencies above its cutoff frequency. This transition occurs in an infinitesimal range of frequencies around the cutoff frequency. One might wonder what happens to a signal at exactly the cutoff frequency. From a practical point of view this question is not significant because the ideal filter cannot be physically realized. No real filter can have vertical sides in its frequency response, but it is interesting from a theoretical point of view to see what would happen if we could build it. We can analyze this situation by convolving the impulse response of an ideal lowpass filter with a cosine and see what happens as we change the frequency of the cosine.

The response of a unity-gain, zero-phase-shift, ideal lowpass filter to a unit-cosine excitation is the convolution of its impulse response with that cosine

$$y(t) = 2f_m \operatorname{sinc}(2f_m t) * \cos(2\pi f_0 t).$$

Using the integral definition of convolution,

$$y(t) = 2f_m \int_{-\infty}^{\infty} \operatorname{sinc}(2f_m \tau) \cos(2\pi f_0(t - \tau)) d\tau$$

or, using the definition of the sinc function,

$$y(t) = \int_{-\infty}^{\infty} \frac{\sin(2\pi f_m \tau)}{\pi \tau} \cos(2\pi f_0(t - \tau)) d\tau.$$

We can use a trigonometric identity for the cosine of a difference angle to write

$$y(t) = \int_{-\infty}^{\infty} \frac{\sin(2\pi f_m \tau)}{\pi \tau} [\cos(2\pi f_0 t) \cos(2\pi f_0 \tau) + \sin(2\pi f_0 t) \sin(2\pi f_0 \tau)] d\tau.$$

The integral of this sum is equal to a sum of two integrals and the second integral is zero because it is the integral of an odd function over symmetrical limits. Therefore,

$$y(t) = \cos(2\pi f_0 t) \int_{-\infty}^{\infty} \frac{\sin(2\pi f_m \tau)}{\pi \tau} \cos(2\pi f_0 \tau) d\tau.$$

Then, using a trigonometric identity for the product of a sin and a cosine,

$$y(t) = \frac{1}{2} \cos(2\pi f_0 t) \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau(f_m - f_0)) + \sin(2\pi\tau(f_m + f_0))}{\pi\tau} d\tau.$$

Now, to evaluate the integral, consider three cases.

Case I. $f_0 < f_m$

$$y(t) = \frac{1}{2} \cos(2\pi f_0 t) \left[2(f_m - f_0) \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau(f_m - f_0))}{2\pi\tau(f_m - f_0)} d\tau + 2(f_m + f_0) \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau(f_m + f_0))}{2\pi\tau(f_m + f_0)} d\tau \right] \quad (\text{J.1})$$

or

$$y(t) = \frac{1}{2} \cos(2\pi f_0 t) \left[\underbrace{2(f_m - f_0) \int_{-\infty}^{\infty} \text{sinc}(2\tau(f_m - f_0)) d\tau}_{=1} + \underbrace{2(f_m + f_0) \int_{-\infty}^{\infty} \text{sinc}(2\tau(f_m + f_0)) d\tau}_{=1} \right]$$

We can use the CTFT to find the areas under these sinc functions and the final result is $y(t) = \cos(2\pi f_0 t)$. In this case, the excitation and response are identical.

Case II. $f_0 = f_m$

In this case, $\sin(2\pi\tau(f_m - f_0)) = 0$ and, from (J.1),

$$y(t) = \frac{1}{2} \cos(2\pi f_0 t) \times \underbrace{2(f_m + f_0) \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau(f_m + f_0))}{2\pi\tau(f_m + f_0)} d\tau}_{=1} = \frac{1}{2} \cos(2\pi f_0 t)$$

In this case the response is exactly half the excitation.

Case III. $f_0 > f_m$

In this case,

$$2(f_m - f_0) \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau(f_m - f_0))}{2\pi\tau(f_m - f_0)} d\tau = 2(f_m - f_0) \int_{-\infty}^{\infty} \text{sinc}(2\tau(f_m - f_0)) d\tau = -1$$

the two integrals in (J.1) exactly cancel and $y(t) = 0$. In this case the response is zero. We can see that, for the ideal lowpass filter, the definition of the rectangle function as having a value of one-half at its discontinuity fits exactly its frequency response.