Web Appendix N - Derivations of the Properties of the LaplaceTransform

N.1 Linearity

Let $\mathrm{z}(t) = \alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t)$ where α and β are constants. Then

$$\mathrm{Z}(\mathrm{s}) = \int_{0^-}^{\infty} \left[\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t)\right] e^{-st}\,dt = \alpha \int_{0^-}^{\infty} \mathrm{x}(t) e^{-st}\,dt + \beta \int_{0^-}^{\infty} \mathrm{y}(t) e^{-st}\,dt = \alpha\,\mathrm{X}(s) + \beta\,\mathrm{Y}(s)$$

and the linearity property is

$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \xleftrightarrow{\mathcal{L}} \alpha\,\mathrm{X}(s) + \beta\,\mathrm{Y}(s) \ .$$

N.2 Time Shifting

Let $\mathrm{z}(t) = \mathrm{g}(t - t_0)$, $t_0 \geq 0$. Then

$$\mathrm{Z}(s) = \int_{0^-}^{\infty} \mathrm{z}(t) e^{-st}\,dt = \int_{0^-}^{\infty} \mathrm{g}(t - t_0) e^{-st}\,dt \ .$$

Let $\tau = t - t_0 \Rightarrow d\tau = dt$. Then

$$\mathrm{Z}(s) = \int_{t_0^-}^{\infty} \mathrm{g}(\tau) e^{-s(\tau - t_0)}\,d\tau = e^{-st_0} \int_{t_0^-}^{\infty} \mathrm{g}(\tau) e^{-s\tau}\,d\tau = e^{-st_0}\,\mathrm{G}(s).$$

N.3 Complex-Frequency Shifting

Let s_0 be a complex constant. Then

$$e^{s_0 t}\,\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \int_{0^-}^{\infty} e^{s_0 t}\,\mathrm{g}(t) e^{-st}\,dt$$

$$e^{s_0 t}\,\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \int_{0^-}^{\infty} \mathrm{g}(t) e^{-(s - s_0)t}\,dt = \mathrm{G}(s - s_0)$$

The complex-frequency-shifting property of the Laplace transform is

$$e^{s_0 t}\,\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s - s_0) \qquad \text{(N.1)}$$

N.4 Time Scaling

Let a be any positive real constant . Then the Laplace transform of $\mathrm{g}(at)$ is

$$\mathrm{g}(at) \xleftrightarrow{\mathcal{L}} \int_{0^-}^{\infty} \mathrm{g}(at) e^{-st}\,dt$$

Let $\tau = at$ and $d\tau = adt$

$$\mathrm{g}(at) \xleftrightarrow{\mathcal{L}} \int_{0^-}^{\infty} \mathrm{g}(\tau) e^{-s\tau/a} \frac{d\tau}{a} = \frac{1}{a}\int_{0^-}^{\infty} \mathrm{g}(\tau) e^{-s\tau/a} d\tau = \frac{1}{a}\mathrm{G}\left(\frac{s}{a}\right) \ , \ a > 0$$

$$\mathrm{g}(at) \xleftrightarrow{\mathcal{L}} (1/a)\mathrm{G}(s/a) \ , \ a > 0 \qquad \text{(N.2)}$$

N.5 Frequency Scaling

Let a be any positive real constant. Then, using the time-scaling property of the Laplace transform $\mathrm{g}(at) \xleftrightarrow{\mathcal{L}} (1/a)\mathrm{G}(s/a)$, $a > 0$. Let $b = 1/a$. Then $\mathrm{g}(t/b) \xleftrightarrow{\mathcal{L}} b\mathrm{G}(bs)$, $b > 0$ or $(1/b)\mathrm{g}(t/b) \xleftrightarrow{\mathcal{L}} \mathrm{G}(bs)$, $b > 0$ and the frequency-scaling property of the Laplace transform is

$$(1/a)\mathrm{g}(t/a) \xleftrightarrow{\mathcal{L}} \mathrm{G}(as) \ , \ a > 0 \qquad \text{(N.3)}$$

N.6 First Time Derivative

The Laplace transform is defined by

$$\mathrm{G}(s) = \int_{0^-}^{\infty} \mathrm{g}(t) e^{-st}\,dt\ .$$

Evaluate the integral by parts using $\int udv = uv - \int vdu$ and let $u = \mathrm{g}(t)$ and $dv = e^{-st}dt$. Then

$$du = \frac{d}{dt}\left(\mathrm{g}(t)\right)dt \quad \text{and} \quad v = -\frac{1}{s}e^{-st}$$

and

$$\int_{0^-}^{\infty} \mathrm{g}(t)e^{-st}dt = \left[\mathrm{g}(t)\left(-\frac{1}{s}\right)e^{-st}\right]_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty}\frac{d}{dt}\left(\mathrm{g}(t)\right)e^{-st}dt$$

$$\mathrm{G}(s) = \frac{1}{s}\mathrm{g}(0^-) + \frac{1}{s}\int_{0^-}^{\infty}\frac{d}{dt}\left(\mathrm{g}(t)\right)e^{-st}dt$$

(where it is understood that $\mathrm{Re}(s) = \sigma$ is chosen to make $\mathrm{G}(s)$ exist). Then

$$\mathcal{L}\left\{\frac{d}{dt}\left(\mathrm{g}(t)\right)\right\} = \int_{0^-}^{\infty}\frac{d}{dt}\left(\mathrm{g}(t)\right)e^{-st}dt = s\mathrm{G}(s) - \mathrm{g}(0^-)$$

and the first-time-derivative property of the Laplace transform is

$$\frac{d}{dt}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} s\mathrm{G}(s) - \mathrm{g}(0^-) \quad . \tag{N.4}$$

N.7 *N*th Time Derivative

This property can be proven using the previous property for the first time derivative and applying it to a first time derivative to form a second time derivative and then generalizing the result to the *N*th time derivative. The second time derivative of a function $\mathrm{g}(t)$ is

$$\frac{d^2}{dt^2}\left(\mathrm{g}(t)\right) = \frac{d}{dt}\left(\frac{d}{dt}\left(\mathrm{g}(t)\right)\right)$$

Therefore, using

$$\frac{d}{dt}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} s\mathrm{G}(s) - \mathrm{g}(0^-)$$

we get

$$\mathcal{L}\left\{\frac{d^2}{dt^2}\left(\mathrm{g}(t)\right)\right\} = s\,\mathcal{L}\left\{\frac{d}{dt}\left(\mathrm{g}(t)\right)\right\} - \left.\frac{d}{dt}\left(\mathrm{g}(t)\right)\right|_{t=0^-}$$

$$\mathcal{L}\left\{\frac{d^2}{dt^2}\left(\mathrm{g}(t)\right)\right\} = s\left\{s\mathrm{G}(s) - \mathrm{g}(0^-)\right\} - \left.\frac{d}{dt}\left(\mathrm{g}(t)\right)\right|_{t=0^-} = s^2\mathrm{G}(s) - s\mathrm{g}(0^-) - \left.\frac{d}{dt}\left(\mathrm{g}(t)\right)\right|_{t=0^-}$$

The second-time-derivative property of the Laplace transform is

$$\frac{d^2}{dt^2}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} s^2\,\mathrm{G}(s) - s\,\mathrm{g}(0^-) - \frac{d}{dt}\left(\mathrm{g}(t)\right)_{t=0^-} .$$

After seeing the derivation of this property from the previous differentiation property we can inductively generalize to the *N*th derivative.

$$\frac{d^N}{dt^N}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} s^N\,\mathrm{G}(s) - \sum_{n=1}^{N} s^{N-n}\left[\frac{d^{n-1}}{dt^{n-1}}\left(\mathrm{g}(t)\right)\right]_{t=0^-} \quad \text{(N.5)}$$

N.8 Complex-Frequency Differentiation

Start with the definition of the Laplace transform

$$\mathrm{G}(s) = \int_{0^-}^{\infty} \mathrm{g}(t)e^{-st}dt .$$

Differentiating with respect to s

$$\frac{d}{ds}\left(\mathrm{G}(s)\right) = \frac{d}{ds}\int_{0^-}^{\infty} \mathrm{g}(t)e^{-st}dt = \int_{0^-}^{\infty} \frac{d}{ds}\left(\mathrm{g}(t)e^{-st}\right)dt = \int_{0^-}^{\infty} -t\,\mathrm{g}(t)e^{-st}dt = \mathcal{L}\left(-t\,\mathrm{g}(t)\right)$$

$$-t\,\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}\left(\mathrm{G}(s)\right) \quad \text{(N.6)}$$

N.9 Multiplication-Convolution Duality

The convolution of $\mathrm{g}(t)$ with $\mathrm{h}(t)$ is

$$\mathrm{g}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{g}(\tau)\mathrm{h}(t-\tau)d\tau .$$

Since $\mathrm{g}(t)$ is zero for time $t < 0$,

$$\mathrm{g}(t) * \mathrm{h}(t) = \int_{0^-}^{\infty} \mathrm{g}(\tau)\mathrm{h}(t-\tau)d\tau .$$

From the definition of the Laplace transform,

$$\mathcal{L}\left[\mathrm{g}(t)*\mathrm{h}(t)\right]=\int_{0^-}^{\infty}\left[\int_{0^-}^{\infty}\mathrm{g}(\tau)\mathrm{h}(t-\tau)d\tau\right]e^{-st}dt$$

$$\mathcal{L}\left[\mathrm{g}(t)*\mathrm{h}(t)\right]=\int_{0^-}^{\infty}\mathrm{g}(\tau)\left[\int_{0^-}^{\infty}e^{-st}\,\mathrm{h}(t-\tau)dt\right]d\tau\,.$$

Since $\mathrm{h}(t)$ is zero for time $t<0$,

$$\mathcal{L}\left[\mathrm{g}(t)*\mathrm{h}(t)\right]=\int_{0^-}^{\infty}\mathrm{g}(\tau)\left[\int_{\tau^-}^{\infty}e^{-st}\,\mathrm{h}(t-\tau)dt\right]d\tau$$

Let $\lambda=t-\tau$ and $d\lambda=dt$. Then

$$\mathcal{L}\left[\mathrm{g}(t)*\mathrm{h}(t)\right]=\int_{0^-}^{\infty}\mathrm{g}(\tau)\left[\int_{0^-}^{\infty}e^{-s(\lambda+\tau)}\,\mathrm{h}(\lambda)d\lambda\right]d\tau$$

$$\mathcal{L}\left[\mathrm{g}(t)*\mathrm{h}(t)\right]=\int_{0^-}^{\infty}e^{-s\tau}\,\mathrm{g}(\tau)\underbrace{\left[\int_{0^-}^{\infty}e^{-s\lambda}\,\mathrm{h}(\lambda)d\lambda\right]}_{\mathrm{H}(s)}d\tau$$

$$\mathcal{L}\left[\mathrm{g}(t)*\mathrm{h}(t)\right]=\mathrm{H}(s)\underbrace{\int_{0^-}^{\infty}e^{-s\tau}\,\mathrm{g}(\tau)d\tau}_{=\mathrm{G}(s)}=\mathrm{G}(s)\mathrm{H}(s)$$

The time-domain convolution property of the Laplace transform is

$$\mathrm{g}(t)*\mathrm{h}(t)\xleftrightarrow{\mathcal{L}}\mathrm{G}(s)\mathrm{H}(s) \qquad \text{(N.7)}$$

The Laplace transform of a product of time-domain functions is

$$\mathcal{L}\left[\mathrm{g}(t)\mathrm{h}(t)\right]=\int_{0^-}^{\infty}\mathrm{g}(t)\mathrm{h}(t)e^{-st}dt$$

$$\mathcal{L}\left[\mathrm{g}(t)\mathrm{h}(t)\right]=\int_{0^-}^{\infty}\left[\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\mathrm{G}(w)e^{wt}dw\right]\mathrm{h}(t)e^{-st}dt$$

where σ is chosen to make $\mathrm{G}(s)$ and $\mathrm{H}(s)$ exist. Doing the *t* integration first,

$$\mathcal{L}\left[\mathrm{g}(t)\mathrm{h}(t)\right] = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\mathrm{G}(w)\left[\int_{0^-}^{\infty}\mathrm{h}(t)e^{-(s-w)t}dt\right]dw.$$

If $\mathrm{H}(s)$ exists then

$$\int_{0^-}^{\infty}\mathrm{h}(t)e^{-(s-w)t}dt = \mathrm{H}(s-w)$$

and

$$\mathcal{L}\left[\mathrm{g}(t)\mathrm{h}(t)\right] = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\mathrm{G}(w)\mathrm{H}(s-w)dw.$$

Therefore

$$\mathrm{g}(t)\mathrm{h}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty}\mathrm{G}(w)\mathrm{H}(s-w)dw \qquad \text{(N.8)}$$

N.10Integration

The integration property is easy to prove, using the convolution property just proven in the last section and the fact that

$$\mathrm{g}(t)*\mathrm{u}(t) = \int_{-\infty}^{\infty}\mathrm{g}(\tau)\mathrm{u}(t-\tau)d\tau = \int_{0^-}^{t}\mathrm{g}(\tau)d\tau$$

$$\mathrm{g}(t)*\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)\mathrm{U}(s) = \mathrm{G}(s)/s$$

Therefore

$$\int_{0^-}^{t}\mathrm{g}(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}\mathrm{G}(s). \qquad \text{(N.9)}$$

N.11Initial Value Theorem

Start with the first-time-derivative property of the Laplace transform

$$\mathcal{L}\left\{\frac{d}{dt}\left(\mathrm{g}(t)\right)\right\} = \int_{0^-}^{\infty}\frac{d}{dt}\left(\mathrm{g}(t)\right)e^{-st}dt = s\mathrm{G}(s) - \mathrm{g}(0^-).$$

Let $s \to \infty$. Then

$$\lim_{s\to\infty}\int_{0^-}^{\infty}\frac{d}{dt}\left(\mathrm{g}(t)\right)e^{-st}dt = \lim_{s\to\infty}\left[s\mathrm{G}(s) - \mathrm{g}(0^-)\right]$$

$$\int_{0^-}^{\infty} \lim_{s\to\infty}\left\{\frac{d}{dt}\big(\mathrm{g}(t)\big)e^{-st}\right\}dt = \lim_{s\to\infty}\left[s\mathrm{G}(s) - \mathrm{g}(0^-)\right].$$

Case I. $\mathrm{g}(t)$ is continuous at $t = 0$

If the Laplace transform of $\mathrm{g}(t)$, which is $\mathrm{G}(s)$, exists for $\mathrm{Re}(s) = \sigma > \sigma_0$, the quantity $\frac{d}{dt}\big(\mathrm{g}(t)\big)e^{-st}$ approaches zero as *s* approaches infinity and

$$0 = \lim_{s\to\infty}\left[s\mathrm{G}(s) - \mathrm{g}(0^-)\right] \Rightarrow \mathrm{g}(0^-) = \lim_{s\to\infty} s\mathrm{G}(s)$$

and, since $\mathrm{g}(t)$ is continuous at $t = 0$, $\mathrm{g}(0^-) = \mathrm{g}(0^+)$ and

$$\mathrm{g}(0^+) = \lim_{s\to\infty} s\mathrm{G}(s).$$

Case II. $\mathrm{g}(t)$ is discontinuous at $t = 0$

In this case, the discontinuity of $\mathrm{g}(t)$ at $t = 0$ means that the derivative of $\mathrm{g}(t)$ has an impulse at $t = 0$ and the strength of the impulse is $\mathrm{g}(0^+) - \mathrm{g}(0^-)$. Now the integral $\lim_{s\to\infty}\int_{0^-}^{\infty}\frac{d}{dt}\big(\mathrm{g}(t)\big)e^{-st}dt$ becomes

$$\lim_{s\to\infty}\int_{0^-}^{\infty}\frac{d}{dt}\big(\mathrm{g}(t)\big)e^{-st}dt = \lim_{s\to\infty}\int_{0^-}^{0^+}\left[\mathrm{g}(0^+) - \mathrm{g}(0^-)\right]\delta(t)e^{-st}dt + \underbrace{\lim_{s\to\infty}\int_{0^+}^{\infty}\frac{d}{dt}\big(\mathrm{g}(t)\big)e^{-st}dt}_{=0}.$$

and, using the sampling property of the impulse in the first integral ,

$$\lim_{s\to\infty}\int_{0^-}^{\infty}\frac{d}{dt}\big(\mathrm{g}(t)\big)e^{-st}dt = \lim_{s\to\infty}\left[\mathrm{g}(0^+) - \mathrm{g}(0^-)\right] = \mathrm{g}(0^+) - \mathrm{g}(0^-)$$

Therefore,

$$\mathrm{g}(0^+) - \mathrm{g}(0^-) = \lim_{s\to\infty}\left[s\mathrm{G}(s) - \mathrm{g}(0^-)\right] = \lim_{s\to\infty} s\mathrm{G}(s) - \mathrm{g}(0^-)$$

or

$$\mathrm{g}(0^+) = \lim_{s\to\infty} s\mathrm{G}(s) \qquad \text{(N.10)}$$

and the result is the same as in Case I.

N.12 Final Value Theorem

From the first-time-derivative property of the Laplace transform,

$$\lim_{s\to 0}\int_{0^-}^{\infty}\frac{d}{dt}\left(\mathrm{g}(t)\right)e^{-st}dt=\lim_{s\to 0}\left[s\mathrm{G}(s)-\mathrm{g}(0^-)\right]$$

$$\int_{0^-}^{\infty}\lim_{s\to 0}\left\{\frac{d}{dt}\left(\mathrm{g}(t)\right)e^{-st}\right\}dt=\lim_{s\to 0}\left[s\mathrm{G}(s)-\mathrm{g}(0^-)\right]$$

$$\int_{0^-}^{\infty}\frac{d}{dt}\left(\mathrm{g}(t)\right)dt=\lim_{s\to 0}\left[s\mathrm{G}(s)-\mathrm{g}(0^-)\right]$$

$$\lim_{t\to\infty}\left[\mathrm{g}(t)-\mathrm{g}(0^-)\right]=\lim_{s\to 0}\left[s\mathrm{G}(s)-\mathrm{g}(0^-)\right]$$

Then, if the limit $\lim_{t\to\infty}\mathrm{g}(t)$ exists, the final value theorem of the Laplace transform is

$$\lim_{t\to\infty}\mathrm{g}(t)=\lim_{s\to 0}s\mathrm{G}(s). \qquad \text{(N.11)}$$