Web Appendix P - Complex Numbers and Complex Functions

P.1 Basic Properties of Complex Numbers

In the history of mathematics there is a progressive broadening of the concept of "numbers". The first numbers were the natural counting numbers, 1,2,3… Next were zero and the negative numbers completing the set we now call *integers*. *Fractions* (ratios of integers) filled in some of the points between integers and later *irrational numbers* filled in all the gaps between fractions to form what we now call the *real* numbers, an infinite continuum of one-dimensional numbers.

In trying to solve quadratic equations of the form $ax^2 + bx + c = 0$ real solutions can always be found if $b^2 - 4ac$ is greater than or equal to zero. But if $b^2 - 4ac$ is less than zero, no real solution can be found. The essence of the problem is in trying to solve the equation $x^2 = -1$ for x. None of the real numbers can be the solution of this equation. The proposal that an *imaginary* number could be the solution to this equation led to a whole new field of mathematics, complex variables. The idea of complex numbers seemed artificial and abstract at first but as mathematical and physical theory has developed, the usefulness of complex numbers solving practical problems has been conclusively shown. The square root of minus one has been given the symbol j and therefore $j^2 = -1$.

Different authors use different symbols to indicate the square root of minus one. A commonly-used symbol is i. This is used in many mathematics and physics books. The symbol j is preferred in most electrical engineering books to avoid confusion because the symbol i is usually reserved for electrical current.

A complex number z can be expressed as the sum of a real number x and an imaginary number jy where y is also a real number. In the complex number $z = x + jy$, x is the real part and y is the imaginary part. (Notice that, although it sounds strange, the <u>imaginary part</u> of a <u>complex</u> number is a <u>real</u> number.) Two complex numbers are equal if, and only if, their real and imaginary parts are equal separately. Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then, if $z_1 = z_2$ that implies that $x_1 = x_2$ and $y_1 = y_2$. In the following material the symbol z will represent some arbitrary complex number and the symbols x and y will represent the real and imaginary parts of z respectively.

The sum and product of two complex numbers are defined as

$$z_1 + z_2 = \left(x_1 + jy_1\right) + \left(x_2 + jy_2\right) = x_1 + x_2 + j\left(y_1 + y_2\right) \tag{P.1}$$

and

$$z_1 z_2 = \left(x_1 + jy_1\right)\left(x_2 + jy_2\right) = x_1 x_2 - y_1 y_2 + j\left(x_1 y_2 + x_2 y_1\right) \quad . \tag{P.2}$$

From (P.1),

$$z + 0 = z + \left(0 + j0\right) = \left(x + 0\right) + j\left(y + 0\right) = x + jy = z$$

proving that the number, zero, is the additive identity for complex numbers just as it is for real numbers. From (P.2),

$$z \times 1 = z \times \left(1 + j0\right) = \left(x\left(1\right) - y\left(0\right)\right) + j\left(x\left(0\right) + y\left(1\right)\right) = x + jy = z$$

proving that the real number 1 is the multiplicative identity for complex numbers, just as it is for real numbers.

By a straightforward extension of the law of addition, subtraction is defined by

$$z_1 - z_2 = x_1 - x_2 + j\left(y_1 - y_2\right).$$

Division can be derived from multiplication and the result is

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} = x_3 + jy_3 = z_3$$

It follows that

$$\frac{z}{z} = 1\,,\ \frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \text{ and } \frac{1}{z_1 z_2} = \frac{1}{z_1}\frac{1}{z_2}\ ,\ \left(z_1 \neq 0\,,\ z_2 \neq 0\right)\ .$$

Example P-1 Adding, subtracting, multiplying and dividing complex numbers with MATLAB

$$\left(3+j2\right)+\left(-1+j6\right)=2+j8 \quad , \quad \left(3+j2\right)-\left(-1+j6\right)=4-j4$$

$$\left(3+j2\right)\left(-1+j6\right)=-15+j16 \quad , \quad \frac{3+j2}{-1+j6}=\frac{9}{37}-j\frac{20}{37}$$

MATLAB handles complex numbers just as easily as real numbers. These four numerical calculations can be done by MATLAB directly at the computer console in the interactive mode. Below is a copy of a MATLAB session doing these calculations.

```
»A = 3 + j*2 ; B = -1+j*6 ;
»A + B
ans =
   2.0000 + 8.0000i
»A - B
ans =
   4.0000 - 4.0000i
»A*B
ans =
 -15.0000 +16.0000i
»A/B
ans =
   0.2432 - 0.5405i
```

The square root of minus one is predefined in MATLAB and is the default value of the variables *i* and *j*.

The commutativity and associativity of complex numbers under addition and multiplication,

$$z_1+z_2=z_2+z_1 \quad , \quad z_1+\left(z_2+z_3\right)=\left(z_1+z_2\right)+z_3 \tag{P.3}$$

and

$$z_1z_2=z_2z_1 \quad , \quad z_1\left(z_2z_3\right)=\left(z_1z_2\right)z_3 \ , \tag{P.4}$$

and the distributivity of complex numbers,

$$z_1\left(z_2+z_3\right)=z_1z_2+z_1z_3 \ , \tag{P.5}$$

can be proven from the definition of complex numbers and the commutativity, associativity and distributivity of real numbers. These properties (P.3), (P.4) and (P.5) lead to the results

$$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \text{ and } \frac{z_1 z_2}{z_3 z_4} = \frac{z_1}{z_3}\frac{z_2}{z_4} \ , \ \left(z_3 \neq 0 \, , \, z_4 \neq 0\right).$$

Just as any real number can be geometrically represented as a point in a one-dimensional space (the real line), any complex number can be represented geometrically by a point in a two-dimensional space, the complex plane (Figure P-1). The complex plane has two orthogonal axes, the real axis and the imaginary axis. Any particular complex number z_0 is defined by its real and imaginary parts x_0 and y_0.

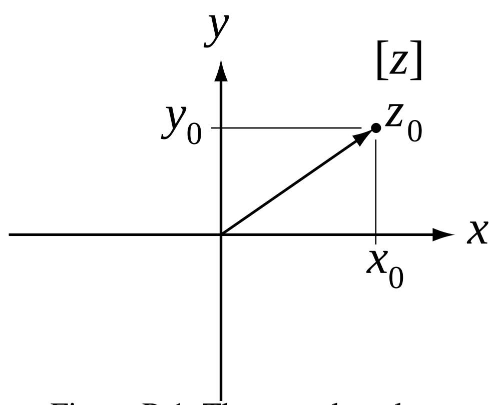

Figure P-1 The complex plane

A vector from the origin of the complex plane to a point can also be used to represent a complex number. The sum and difference of two complex numbers can be found by the usual rules of vector addition and subtraction. In Figure P-2 are two examples of the addition of two complex numbers.

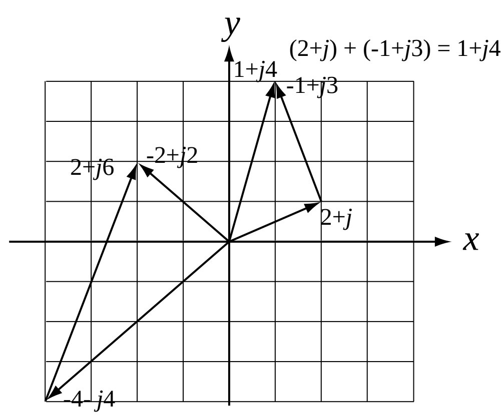

Figure P-2 Graphical addition of complex numbers by vector addition

The complex conjugate of a complex number is found by negating its imaginary part. It is indicated by the addition of a superscript asterisk "*" to the number. If $z_0 = x_0 + jy_0$ the complex conjugate of z_0 is $z_0^* = x_0 - jy_0$. The complex conjugate of a complex number is its reflection in the real axis of the complex plane (Figure P-3).

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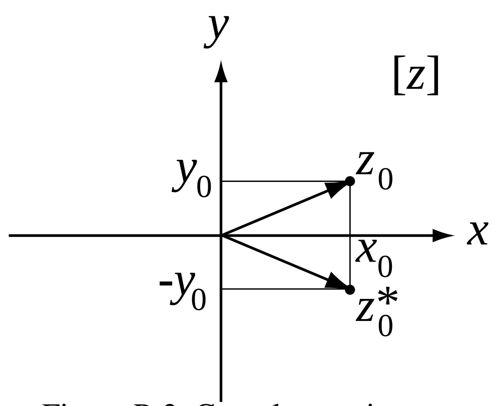


Figure P-3 Complex conjugates

Some properties of conjugates that can be derived from earlier properties of complex numbers are

$$\left(z_1 + z_2\right)^* = z_1^* + z_2^* \,, \left(z_1 - z_2\right)^* = z_1^* - z_2^* \,, \left(z_1 z_2\right)^* = z_1^* z_2^* \,, \left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$$

Also the sum of any complex number and its conjugate is real, and the difference between any complex number and its conjugate is imaginary.

The *absolute value* $|z|$ (or *magnitude* or *modulus*) of a complex number, $z = x + jy$, is the length of the vector in the complex plane which represents z, which is (from the Pythagorean theorem) $|z| = \sqrt{x^2 + y^2}$.

Pythagoras of Samos, 569 BC – 475 BC

By extension, the distance between any two complex numbers z_1 and z_2 in the complex plane is

$$|z_1 - z_2| = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2} \,.$$

The magnitude of a complex number is a real number $|z| = \sqrt{x^2 + y^2} + j0$. A handy relation in the study of complex variables and functions of a complex variable is $|z|^2 = zz^* = x^2 + y^2$. Also,

$$|z_1 z_2| = |z_1||z_2| \quad , \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} .$$

P.2 The Polar Form

It is often convenient in analysis to represent a complex number in polar form. Instead of specifying its real and imaginary parts, we specify its magnitude r and the angle θ that its vector representation in the complex plane makes with the positive real axis, with the counter-clockwise direction being positive. The relations are

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta) \text{ and } z = r\left[\cos(\theta) + j\sin(\theta)\right].$$

The length of the vector r is the magnitude of the complex number $r = |z|$ and the angle or phase θ is related to x and y by $\tan(\theta) = y/x$ (Figure P-4).

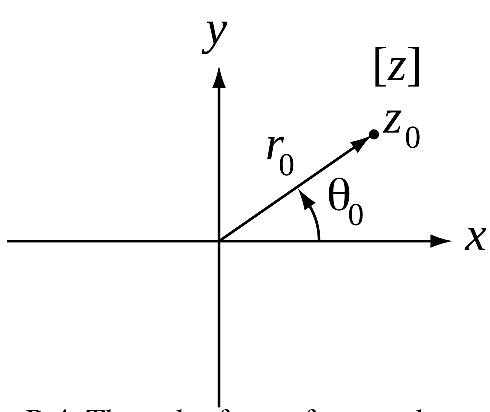


Figure P-4 The polar form of a complex number

There is more than one value of θ that satisfies $\tan(\theta) = y/x$, therefore the angle or phase of a complex number is *multiple valued.* If θ is a solution, so is $\theta + 2n\pi$ where n is any integer. One special case is worthy of note; the case $x = y = 0$. In this case, the ratio y/x is undefined. That means that $\tan(\theta) = y/x$ and, by implication, θ are also undefined. The phase of a complex number whose magnitude is zero is undefined. This should not be cause for alarm. If the magnitude is zero, the vector from the origin of the complex plane to the complex number is a zero-length vector, or a point, the origin. Geometrically the angle from the positive real axis to this vector has no meaning because the vector has no length and therefore, no direction. Also, since the real and imaginary parts of the complex number are found from $x = r\cos(\theta)$ and $y = r\sin(\theta)$, if r is zero, x

and y are also zero regardless of the value of θ. So the mathematics is telling us something logical (as it usually does). If the magnitude is zero, phase has no meaning!

The product of two complex numbers, written in polar form, is

$$z_1 z_2 = r_1\left[\cos\left(\theta_1\right) + j\sin\left(\theta_1\right)\right] r_2\left[\cos\left(\theta_2\right) + j\sin\left(\theta_2\right)\right]$$

$$z_1 z_2 = r_1 r_2\left\{\cos\left(\theta_1\right)\cos\left(\theta_2\right) - \sin\left(\theta_1\right)\sin\left(\theta_2\right) + j\left[\cos\left(\theta_1\right)\sin\left(\theta_2\right) + \sin\left(\theta_1\right)\cos\left(\theta_2\right)\right]\right\}$$

$$z_1 z_2 = \frac{r_1 r_2}{2}\left\{\begin{array}{l}\cos\left(\theta_1 - \theta_2\right) + \cos\left(\theta_1 + \theta_2\right) - \left[\cos\left(\theta_1 - \theta_2\right) - \cos\left(\theta_1 + \theta_2\right)\right] \\ + j\left[\sin\left(\theta_2 - \theta_1\right) + \sin\left(\theta_2 + \theta_1\right) + \sin\left(\theta_1 - \theta_2\right) + \sin\left(\theta_1 + \theta_2\right)\right]\end{array}\right\}$$

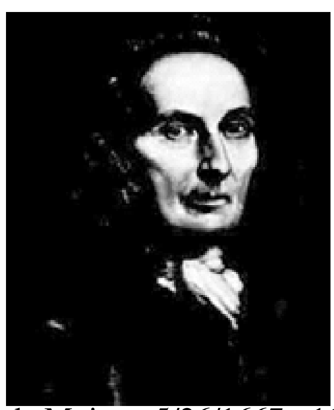
$$z_1 z_2 = r_1 r_2\left[\cos\left(\theta_1 + \theta_2\right) + j\sin\left(\theta_2 + \theta_1\right)\right]$$

The *magnitude of the product* of two complex numbers is the *product* of their magnitudes and the *angle of the product* of two complex numbers is the *sum* of their angles. Applying this idea to the product of multiple complex numbers leads to De Moivre's theorem

$$z^n = r^n\left[\cos\left(n\theta\right) + j\sin\left(n\theta\right)\right].$$

It also follows that the *magnitude of the quotient* of two complex numbers is the *quotient* of their magnitudes and the *angle of the quotient* of two complex numbers is the *difference* of their angles

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + j\sin\left(\theta_1 - \theta_2\right)\right] \;,\; r_2 \neq 0\,.$$



Abraham de Moivre, 5/26/1667 - 11/27/1754

Example P-2 Polar-to-rectangular and rectangular-to-polar number conversions

$$9+j7=11.4\left[\cos\left(0.661\right)+j\sin\left(0.661\right)\right] \text{ or } 11.4\measuredangle 0.661 \text{ or } 11.4\measuredangle 37.87°$$

$$-4+j8=8.94\left[\cos\left(2.034+j\sin\left(2.034\right)\right)\right] \text{ or } 8.94\measuredangle 2.034 \text{ or } 8.94\measuredangle 116.57°$$

$$\left(9+j7\right)\left(-4+j8\right)=11.4\left[\cos\left(0.661\right)+j\sin\left(0.661\right)\right]\times 8.94\left[\cos\left(2.034\right)+j\sin\left(2.034\right)\right]$$

$$\left(9+j7\right)\left(-4+j8\right)=11.4\times 8.94\left[\cos\left(0.661+2.034\right)+j\sin\left(0.661+2.034\right)\right]$$

$$\left(9+j7\right)\left(-4+j8\right)=101.98\left[\cos\left(2.695\right)+j\sin\left(2.695\right)\right]=-92+j44$$

$$\frac{9+j7}{-4+j8}=\frac{11.4\left[\cos\left(0.661\right)+j\sin\left(0.661\right)\right]}{8.94\left[\cos\left(2.034\right)+j\sin\left(2.034\right)\right]}=\frac{11.4}{8.94}\left[\cos\left(0.661-2.034\right)+j\sin\left(0.661-2.034\right)\right]$$

$$\frac{9+j7}{-4+j8}=1.275\left[\cos\left(-1.373\right)+j\sin\left(-1.373\right)\right]=\frac{1}{4}-j\frac{5}{4}$$

In MATLAB,

```
»A = 9+j*7 ; B = -4+j*8 ;
»abs(A)
ans =
   11.4018
»angle(A)
ans =
    0.6610
»abs(B)
ans =
    8.9443
»angle(B)
ans =
    2.0344
»A*B
ans =
 -92.0000 +44.0000i
»abs(A*B)
ans =
  101.9804
»angle(A*B)
ans =
    2.6955
»A/B
ans =
   0.2500 - 1.2500i
```

```
»abs(A/B)
ans =
    1.2748
»angle(A/B)
ans =
   -1.3734
```

The nth root z_o of a complex number z is the solution of the equation $z_0^n = z$ where n is a positive integer. In polar form,

$$z = r\left[\cos(\theta) + j\sin(\theta)\right] \text{ and } z_0 = r_0\left[\cos(\theta_0) + j\sin(\theta_0)\right].$$

Then

$$r_0^n\left[\cos(n\theta_0) + j\sin(n\theta_0)\right] = r\left[\cos(\theta) + j\sin(\theta)\right]$$

and the solutions of for r and θ_0 are

$$r_0 = \sqrt[n]{r} \text{ and } \theta_0 = \frac{\theta + 2k\pi}{n},\ k \text{ an integer,}$$

and there are exactly n *distinct* values. The n distinct nth roots of the real number, +1, are

$$1^{1/n} = \cos(2k\pi / n) + j\sin(2k\pi / n)\ ,\ k = 0,1,\ldots,n-1.$$

Notice that each of the nth roots of any complex number lies on a circle in the complex plane. The circle is centered at the origin, the radius of the circle is the positive real nth root of the magnitude of the complex number and the n nth roots are spaced at equal angular intervals of $2\pi / n$ radians (Figure P-5). Therefore, in any problem of finding roots of a complex number, if we can find one root, the others are easily found by putting them in a symmetrical array of complex numbers with the same magnitude and the proper angular spacing. Usually the easiest root to find first is the one whose magnitude is the positive real nth root of the magnitude of the complex number and whose angle is the angle of the complex number, divided by n.

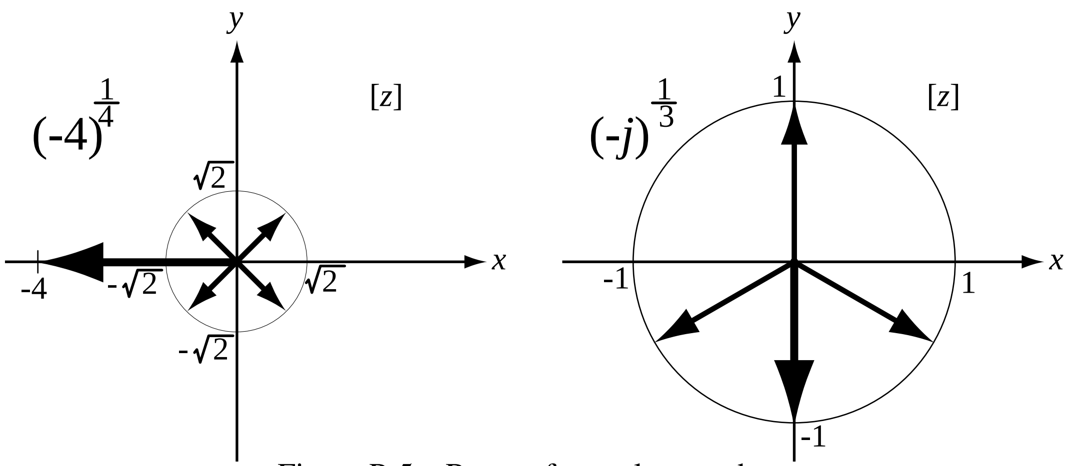


Figure P-5 Roots of complex numbers

MATLAB has a function for finding the roots of an equation `roots`. If the equation is of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots a_2 z^2 + a_1 z + a_0 = 0 \ , \tag{P.6}$$

then the MATLAB command `roots(`[a_n a_{n-1} $\cdots a_2$ a_1 a_0]`)` or `roots(`[$a_n, a_{n-1}, \cdots a_2, a_1, a_0$]`)` or `roots(`[$a_n; a_{n-1}; \cdots a_2; a_1; a_0$]`)` returns all the distinct roots of (P.6). For example,

```
»roots([3 2])

ans =

  -0.6667
```

or

```
»roots([9,1,-3,6])

ans =

 -1.0432
  0.4661 + 0.6495i
 0.4661 - 0.6495i
```

or

```
»roots([2 ; j*3 ; 1 ; -9])

ans =

-0.6573 - 2.1860i
-0.7464 + 1.1162i
```

1.4037 - 0.4301i

P.3 Functions of a Complex Variable

In the study of signals and systems, probably the most important of the functions of a complex variable is the exponential function, defined as

$$\exp(z) = e^x\left[\cos(y) + j\sin(y)\right] . \tag{P.7}$$

Common nomenclature in signal and system analysis is that the exponential function of a complex variable is called a *complex exponential*. Note that if $y = 0$ in (P.7), z becomes real and this definition collapses to the more familiar definition of the exponential function for real variables $\exp(z) = \exp(x) = e^x$. If z is purely imaginary in (P.7) $x = 0$ and

$$\exp(jy) = \cos(y) + j\sin(y) .$$

This is known as Euler's (pronounced "oilers") identity after Leonhard Euler one of the great early mathematicians.

Leonhard Euler, 4/15/1707 - 9/18/1783

The most common occurrences of the complex exponentials in signal and system theory are with either time t or cyclic frequency f or radian frequency ω as the independent variable, for example

$$\mathrm{x}(t) = 24e^{(\sigma_0 - j\omega_0)t} \quad \text{or} \quad \mathrm{X}(f) = 5e^{-j2\pi f t_0} \quad \text{or} \quad \mathrm{X}(j\omega) = -3e^{j4\omega}$$

where σ_o, ω_0 and t_0 are real constants. When the argument of the exponential function is purely imaginary, the resulting complex exponential is called a *complex sinusoid* because it contains a cosine and a sine as its real and imaginary parts as illustrated in Figure P-6 for a complex sinusoid in time. The projection of the complex sinusoid onto a plane parallel to the plane containing the real and t axes is the cosine function and the projection onto a plane parallel to the plane containing the imaginary and t axes is the sine function.

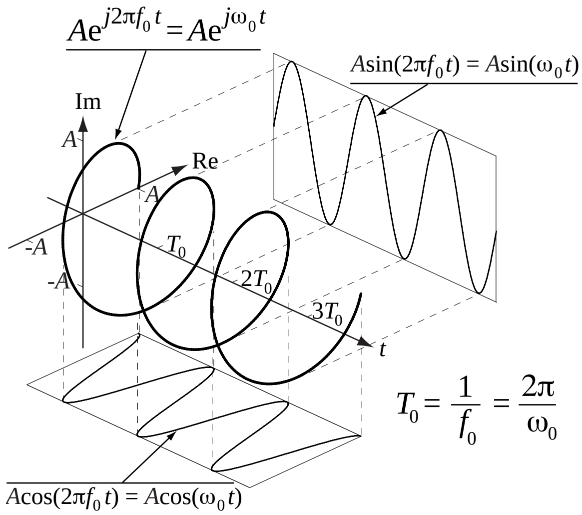


Figure P-6 Relation between a complex sinusoid and a real sine and a real cosine

Other important properties of the exponential function are

$$\frac{\exp(z_1)}{\exp(z_2)} = \exp(z_1 - z_2) \quad , \quad \left[\exp(z)\right]^n = \exp(nz)$$

$$\left[\exp(z)\right]^{\frac{m}{n}} = \exp\left[\frac{m}{n}(z + j2k\pi)\right], k = 0,1,\cdots n-1$$

The exponential function is periodic with period $j2\pi$. That is, it is periodic <u>in the imaginary dimension</u>. This is shown from the definition, (P.7), by substituting $z + j2n\pi$ for z

$$\exp(z + j2n\pi) = e^x\left[\cos(y + 2n\pi) + j\sin(y + 2n\pi)\right] = e^x\left[\cos(y) + j\sin(y)\right] = \exp(z)$$

n an integer. Also $\exp(z^*) = \left[\exp(z)\right]^*$. Lastly, if a particular complex number z is represented by the polar form $z = r\left[\cos(\theta) + j\sin(\theta)\right]$ then, from Euler's identity one can write $z = r\exp(j\theta) = re^{j\theta}$ which is a convenient way of representing a complex number in many types of analysis. From Euler's identity one can form $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$. Adding $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$,

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta) \Rightarrow \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

Similarly,

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j2}.$$

These two results are important because they show again the intimate relationship between sines, cosines and complex exponentials. It is important to note that if θ is a real number, then the function $\cos(\theta)$ is real-valued. But the equality

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

expresses this <u>real –valued</u> function of a <u>real</u> variable in terms of a combination of <u>complex-valued</u> functions. This only works because $e^{j\theta}$ and $e^{-j\theta}$ are complex conjugates (for real θ) and when a number is added to its complex conjugate the sum is real.

Other properties of trigonometric functions of a complex variable are summarized below:

$$\frac{d}{dz}(\sin(z)) = \cos(z) \quad \text{and} \quad \frac{d}{dz}(\cos(z)) = -\sin(z)$$

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

$$\cos(z) = \frac{e^{y} + e^{-y}}{2}\cos(x) - j\frac{e^{y} - e^{-y}}{2}\sin(x)$$

$$\sin^2(z) + \cos^2(z) = 1$$

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$$

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$$

$$\sin(-z) = -\sin(z) \quad \text{and} \quad \cos(-z) = \cos(z)$$

$$\sin\left(\frac{\pi}{2} - z\right) = \cos(z)$$

$$\sin(2z) = 2\sin(z)\cos(z) \quad \text{and} \quad \cos(2z) = \cos^2(z) - \sin^2(z)$$

MATLAB implements all the exponential and trigonometric functions. The exponential function is `exp`, the sine function is `sin`, the cosine function is `cos`, the tangent function is `tan`, etc... In the trigonometric functions, the argument is always interpreted as an angle in radians. For example,

```
»exp(1)

ans =

    2.7183

»exp(-j*pi)

ans =

    -1

»cos(3*pi/4)

ans =

   -0.7071

»tan(-pi/4)

ans =

   -1.0000
```

P.4 Complex Functions of a Real Variable

In transform analysis there are many examples of complex functions of a real variable. Since the function value is complex, it cannot be as simply graphed as a single plot of a real function of a real variable. There are several methods of plotting functions like this and each has its advantages and disadvantages. We can plot the real and imaginary parts separately as functions of the real independent variable or plot the magnitude and phase separately as functions of the real independent variable or plot the real and imaginary parts versus the independent variable in one three-dimensional isometric plot. As an illustrative example, suppose we want to plot the function $\mathrm{x}(t) = e^{j2\pi t}$. Figure P-7 through Figure P-9 illustrate the three types of plots of this function.

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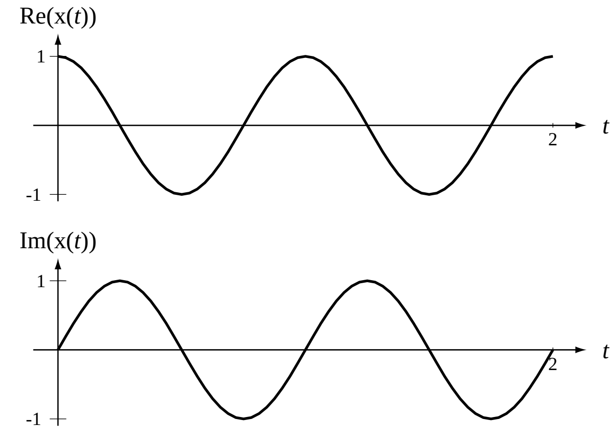


Figure P-7 Real and imaginary parts plotted separately versus the independent variable

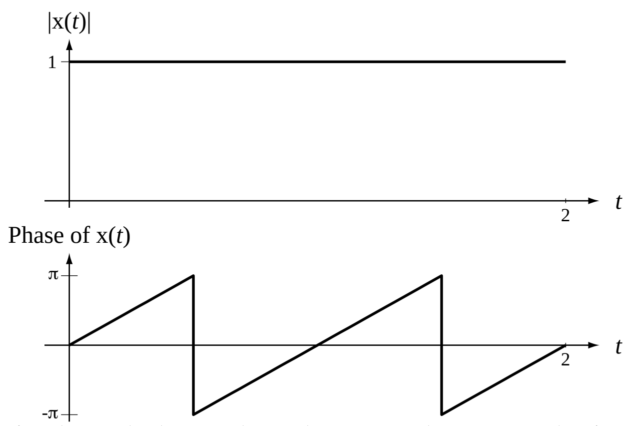


Figure P-8   Magnitude and phase plotted separately versus the independent variable

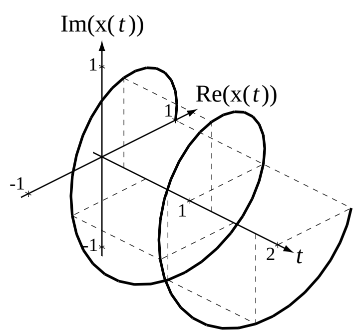


Figure P-9 Real and imaginary parts plotted together versus the independent variable in a three-dimensional isometric plot

Although plots of the real and imaginary parts are sometimes useful, for most analysis purposes, separate plots of the magnitude and phase of a complex function of a real variable are preferred. In the study of transform methods, the independent variable will often be frequency f or ω instead of time t.

Consider a complex function of radian frequency ω

$$\mathrm{H}(j\omega) = \frac{1}{1+j\omega}.$$

How would we plot its magnitude and phase? The square of the magnitude of any complex number is the product of the number and its complex conjugate. Therefore the magnitude of $\mathrm{H}(j\omega)$ is $|\mathrm{H}(j\omega)| = \sqrt{\mathrm{H}(j\omega)\mathrm{H}^*(j\omega)}$. In this case

$$|\mathrm{H}(j\omega)| = \sqrt{\frac{1}{1+j\omega}\frac{1}{1-j\omega}} = \sqrt{\frac{1}{1+\omega^2}} = \frac{1}{\sqrt{1+\omega^2}}.$$

The phase of a complex number is the inverse tangent of the ratio of its imaginary part to its real part. The real and imaginary parts of $\mathrm{H}(j\omega)$ are

$$\mathrm{Re}(\mathrm{H}(j\omega)) = \mathrm{Re}\left(\frac{1}{1+j\omega}\frac{1-j\omega}{1-j\omega}\right) = \mathrm{Re}\left(\frac{1-j\omega}{1+\omega^2}\right) = \frac{1}{1+\omega^2}$$

and

$$\mathrm{Im}(\mathrm{H}(j\omega)) = \mathrm{Im}\left(\frac{1}{1+j\omega}\frac{1-j\omega}{1-j\omega}\right) = \mathrm{Im}\left(\frac{1-j\omega}{1+\omega^2}\right) = -\frac{j\omega}{1+\omega^2}.$$

Therefore the phase of $\mathrm{H}(j\omega)$ is

$$\measuredangle\mathrm{H}(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}(\mathrm{H}(j\omega))}{\mathrm{Re}(\mathrm{H}(j\omega))}\right) = \tan^{-1}\left(\frac{-\frac{j\omega}{1+\omega^2}}{\frac{1}{1+\omega^2}}\right) = \tan^{-1}(-j\omega).$$

It should be noted here that the inverse tangent function is <u>multiple-valued</u>. This means that, strictly speaking, there is a <u>countable infinity</u> of correct values of $\measuredangle\mathrm{H}(j\omega)$ at any arbitrary value of ω. (There is also an <u>uncountable infinity of incorrect</u> values!) If θ is any correct value of $\measuredangle\mathrm{H}(j\omega)$, then $\theta + 2n\pi$, n any integer, is also a correct value of θ because the sines of θ and $\theta + 2n\pi$ are identical and the cosines of θ and $\theta + 2n\pi$ are identical and therefore the real part of $\mathrm{H}(j\omega)\,|\mathrm{H}(j\omega)|\cos(\theta + 2n\pi)$ is the same for any integer value of n and the imaginary part of $\mathrm{H}(j\omega)$ $|\mathrm{H}(j\omega)|\sin(\theta + 2n\pi)$ is also the same for any integer value of n. To avoid any needless confusion caused by the multiple-valued nature of the inverse tangent function it is conventional to restrict plots of phase to lie in some range of angles for which the inverse tangent function is single-valued, for example, $-\pi < \theta \le \pi$. This simply means that when we evaluate the inverse tangent function we choose a correct value that lies in that range. Since any correct phase is as good as any other this causes no problems. Using this convention, the magnitude and phase of $\mathrm{H}(j\omega)$ versus frequency are illustrated in Figure P-10.

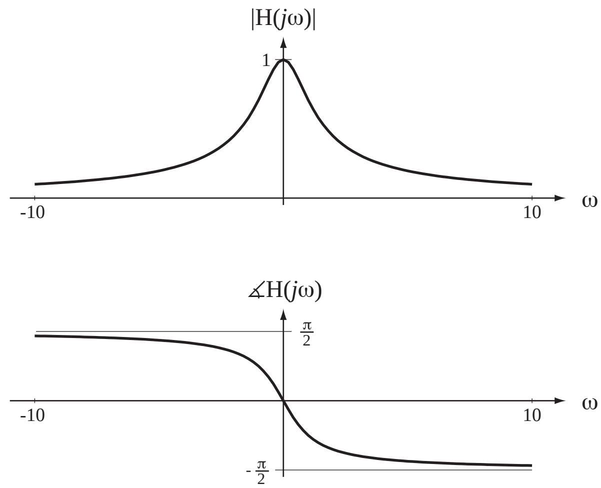


Figure P-10 Magnitude and phase of $\mathrm{H}(j\omega) = \dfrac{1}{1+j\omega}$

These plots were made using MATLAB so they would be very accurate. But it is important to develop quick approximate methods to visualize and sketch the magnitude and phase of complex functions of a real variable. This is a skill that helps an engineer in the design and analysis of systems. Look again at the function

$$\mathrm{H}(j\omega) = \frac{1}{1+j\omega} .$$

We can get a very good quick indication of the general shape of the magnitude and phase by finding the magnitude and phase at some extreme points, ω approaching zero from above or below and ω approaching plus or minus infinity.

For ω equal to zero, the denominator $1+j\omega$ of $\mathrm{H}(j\omega)$ is simply 1 and $\mathrm{H}(j\omega)$ obviously equals one, the real number one whose magnitude is one and whose phase is zero. For ω approaching zero from above (from positive values) the phase of $\mathrm{H}(j\omega)$ is the phase of the numerator (which is zero) minus the phase of the denominator. The phase of the denominator for small positive ω is a small positive phase. Therefore the phase of $\mathrm{H}(j\omega)$ is zero minus a small positive phase, that is, a small negative phase. This shows that as ω approaches zero from above the phase is negative and approaching zero. By similar reasoning as ω approaches zero from below (from negative values), the phase is positive and approaching zero. This analysis is confirmed by the phase plot in Figure P-10.

As ω approaches positive infinity, the denominator $1+j\omega$ becomes infinite in magnitude and, since the numerator is finite, the magnitude of $\mathrm{H}(j\omega)$ approaches zero. Also the 1 in the denominator $1+j\omega$ of $\mathrm{H}(j\omega)$ becomes negligible in comparison with $j\omega$ and the phase of $\mathrm{H}(j\omega)$ approaches zero minus $\pi/2$ which is $-\pi/2$ radians. As ω approaches negative infinity, the phase of $\mathrm{H}(j\omega)$ approaches zero minus $-\pi/2$ which is $\pi/2$ radians. These limits are also confirmed by Figure P-10. For a function as simple as this example, we can sketch a fairly accurate magnitu$1-f^2+jf$ de and phase plot very quickly just using these simple principles.

Now let's try a somewhat more complicated example, a complex function of cyclic frequency *f*

$$\mathrm{H}(f)=\frac{1-f^2}{1-f^2+jf}.$$

Using the quick-approximation ideas just presented, at $f=0$ $\mathrm{H}(f)$ is 1. For *f* approaching zero from above, the phase is the phase of $1-f^2$, which for small *f* is zero, minus the phase of $1-f^2+jf$, which for small positive *f* is a small positive phase. Therefore the phase for *f* approaching zero from above is a small negative phase approaching zero. Similarly for *f* approaching zero from below the phase is a small positive phase approaching zero.

For *f* approaching either positive or negative infinity, the f^2 terms in the numerator $1-f^2$ and in the denominator dominate and the ratio of numerator to denominator approaches $(-f^2)/(-f^2)$ which is one. So the magnitude approaches one and the phase approaches zero in that limit. So far we see that for very small or very large values of *f*, the magnitude approaches one and the phase approaches zero. We might be inclined to assume that the magnitude is one for all frequencies. But consider the case $f=\pm1$. At those values of *f*, the magnitude of $\mathrm{H}(f)$ is zero. Therefore the magnitude must begin at one for $f=0$, go to zero at $f=\pm1$ and approach one for *f* approaching $\pm\infty$. Also, as *f* approaches $+1$ from below, the numerator is a small positive real number with a phase of zero, and the denominator is a small positive number plus an imaginary number approaching *j*. The denominator phase is approaching $\pi/2$ so the phase of $\mathrm{H}(f)$ is approaching $-\pi/2$. As *f* approaches zero from above, the numerator is a small *negative* real number with a phase of π, the phase of the denominator approaches $\pi/2$ and the phase of $\mathrm{H}(f)$ approaches $\pi/2$. So as *f* moves from just below $+1$ to just above $+1$, the phase changes *discontinuously* from $-\pi/2$ to $\pi/2$. Notice that this discontinuity in the phase occurs where the magnitude is exactly zero. Figure P-11 is a plot generated by MATLAB of the magnitude and phase of $\mathrm{H}(f)$ and it confirms all these observations about the magnitude and phase.

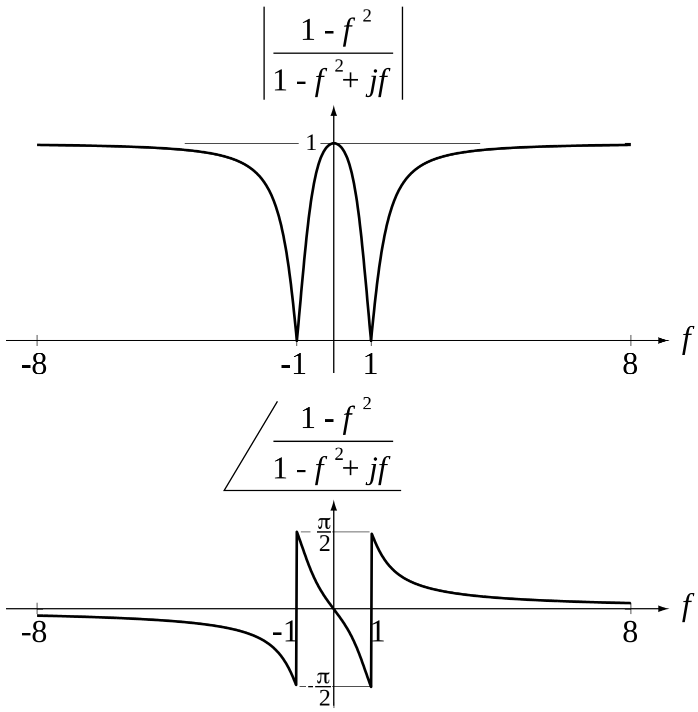


Figure P-11 Magnitude and phase of a complex function of a real frequency

We have already explored the multiple-valued nature of the inverse tangent function. There is one more wrinkle in the computation of phase that is important. We will illustrate it by finding the phase of the complex number $z = -1 + j$. If we take a simple direct approach using a hand-held calculator we might calculate the phase as

$$\text{Phase of } z = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

However a plot of z in the complex plane (Figure P-12) shows that this answer is wrong.

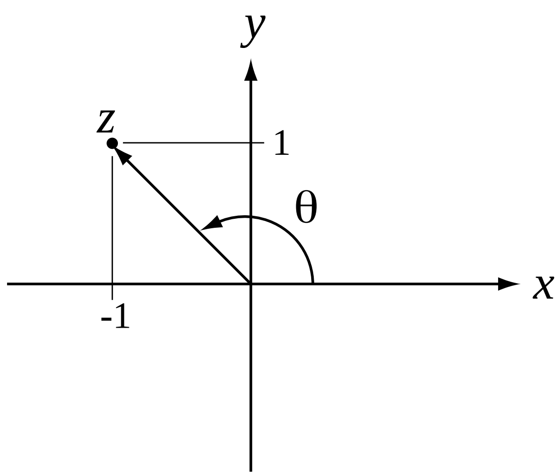


Figure P-12 Location of $-1+j$ in the complex plane

The plot shows that z lies in the second quadrant. The calculator result indicates that z lies in the fourth quadrant. Instead of simply evaluating the inverse tangent of a complex number or function we should evaluate the *four-quadrant* inverse tangent using our knowledge of the real and imaginary parts separately instead of knowledge of their ratio alone. This enables us to locate the quadrant in which the number lies and eliminate a false answer which lies at π radians from the correct answer in the diagonally-opposite quadrant. The problem in using the simple inverse tangent function without thinking is that

$$\measuredangle(-1+j) = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

and

$$\measuredangle(1-j) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

A hand-held calculator typically returns a value θ using the inverse tangent function, in the range $-\pi/2 < \theta \le \pi/2$. The exact location of the complex number in the complex plane is lost when the ratio of the imaginary to real part is taken. Therefore any four-quadrant inverse tangent must take two arguments, the real and imaginary parts separately, rather than their ratio. (MATLAB has a function `angle` which finds the four-quadrant angle or phase of a complex number.) Using a four-quadrant inverse tangent, the correct answer in this example would be

$$\measuredangle(-1+j) = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}.$$

What is the phase of the real number -1? Using the four quadrant tangent, the answer is π (plus or minus any integer multiple of 2π). Therefore when plotting the magnitude and phase of a real-valued function the magnitude is always non-negative and the phase switches back and forth between zero and π (or $-\pi$) as the function values go through zero. Of course, plotting the magnitude and phase of a real function is a little silly since it can be plotted with positive and negative values on a single plot. But, as illustrated above, as soon as the function becomes complex, magnitude and phase plots are one of the best ways to graphically represent the function.

Example P-3 Magnitude and phase of a complex function of a real variable

Using MATLAB, plot the magnitude and phase of the function,

$$X(f) = \frac{1 - f^2}{1 - f^2 + jf}$$

versus *f*, which appears in Figure P-11.

This function can be easily plotted using the `fplot` command in MATLAB. The following sequence of MATLAB commands produces the plot of the magnitude and phase of this function in Figure P-13.

```
subplot(2,1,1) ; fplot('abs((1-f^2)/(1-f^2+j*f))',[-8,8],'k') ;
xlabel('Frequency, f (Hz)') ; ylabel('|X(f)|') ;
subplot(2,1,2) ; fplot('angle((1-f^2)/(1-f^2+j*f))',[-8,8],'k') ;
xlabel('Frequency, f (Hz)') ; ylabel('Phase of X(f)') ;
```

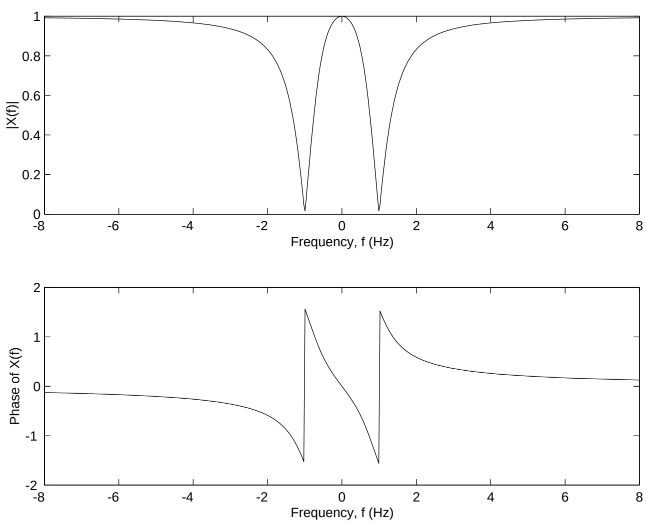


Figure P-13 Magnitude and phase of $X(f) = \frac{1 - f^2}{1 - f^2 + jf}$

Although this plot is easy to generate it does not allow as much user control over formatting and scaling as a somewhat more involved plotting technique. We are plotting a continuous function of the variable *f*. MATLAB does not know what a “continuous” function is. MATLAB can only draw straight lines. Therefore we must formulate the problem so as to get a plot that looks like the continuous function using numerical calculations and plotting with straight lines (Figure P-14).

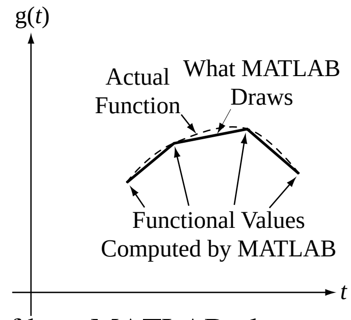


Figure P-14 Illustration of how MATLAB plots an approximation to a function

When we use `fplot` MATLAB decides how to assign the values of the independent variable at which the function values will be calculated. Although the algorithm used by MATLAB is generally very good we can have more control over the plotting of the function and formatting of the plot if we generate the independent variable values ourselves and use the `plot` command instead. The `plot` command is more primitive than the `fplot` command but it also allows the programmer more options in plotting.

In order to get a smooth looking curve we must be sure that the points in f are close enough together that when we draw straight lines between points the result looks like the actual curved contours of $X(f)$. The plot in Figure P-11 covers the range $-8 < f < 8$. How many points do we need to make the plot look smooth? The main requirement on how close the points should be is to resolve the region around $f = \pm 1$ where there are some sharp corners in the function's magnitude and phase. Let's try a spacing between points of $1/10$. The MATLAB program then might look like the following code.

```
%       Program to plot the function, (1-f^2)/(1-f^2+j*f)

%-----------------------------------------------------------------
%       This section actually calculates values of the function
%-----------------------------------------------------------------

df = 1/10 ;                 %  "df" - spacing between frequencies
fmin = -8 ; fmax = 8 ;             %  "fmin" & "fmax" - beginning and ending
                            %    frequencies
f = fmin:df:fmax ;          %  "f" - vector of frequencies for
                                   %    plotting function with straight lines
                            %    between points
X = (1-f.^2)./(1-f.^2+j*f) ;       %  "X" - vector of function values

%-----------------------------------------------------------------
%       This section displays the results and formats the plots
%-----------------------------------------------------------------

subplot(2,1,1) ;                   %  Plot two plots, one on top and one on
                                   %    the bottom.
                                   %  First draw the top plot.
p = plot(f,abs(X),'k') ;    %  Plot |X(f)| with black lines between
                                   %    points
```

```
set(p,'LineWidth',2) ;           %  Make the plot line heavier
xlabel('Frequency, f (Hz)') ;    %  Label the "f" axis
ylabel('|X(f)|') ;       %  Label the "|X(f)|" axis
title('Plot of (1-f.^2)./(1-f.^2+j*f)') ; %  Title the plots
subplot(2,1,2) ;                 %  Draw the second plot.
p = plot(f,angle(X),'k') ; %  Plot the phase (angle) of X(f) with
                                 %    black lines between points
set(p,'LineWidth',2) ;           %  Make the plot line heavier
xlabel('Frequency, f (Hz)') ;    %  Label the "f" axis
ylabel('Phase of X(f)') ;  %  Label the "Phase of X(f)" axis
```

The actual MATLAB graph is displayed in Figure P-15.

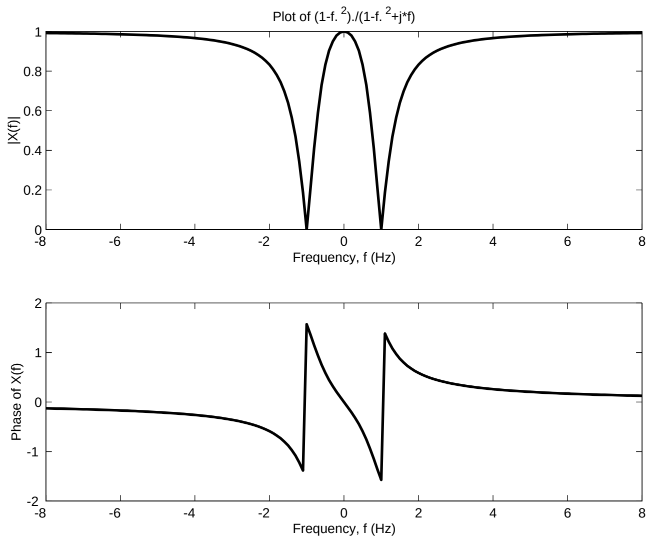


Figure P-15 MATLAB plots of the magnitude and phase of $\mathrm{X}(f) = \dfrac{1-f^2}{1-f^2+jf}$

Although this plot looks very much like Figure P-11, it is not exactly the same. The jumps in the phase plot are not as nearly vertical as in Figure P-11. That is because the spacing between points is not quite small enough. Figure P-16 is the phase plot, redone with the point locations indicated by small dots.

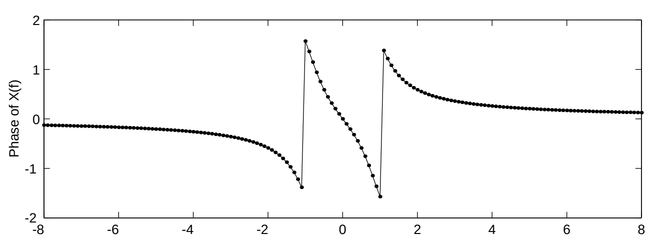


Figure P-16 MATLAB plot of the phase emphasizing the points at which phase is actually calculated

Try a smaller spacing and see what your plot looks like. (This is our first consideration of the sampling problem of representing a continuous function by discrete samples.)

Why is there a jump in the phase anyway? Does the phase of $X(f)$ really change discontinuously at $f = \pm 1$? Notice that the size of the jump is exactly π radians. One way to grasp what is really happening near $f = \pm 1$ is to graph the imaginary part of $X(f)$ versus the real part of $X(f)$ in the complex plane, for a succession of f's near $f = 1$ (Figure P-17).

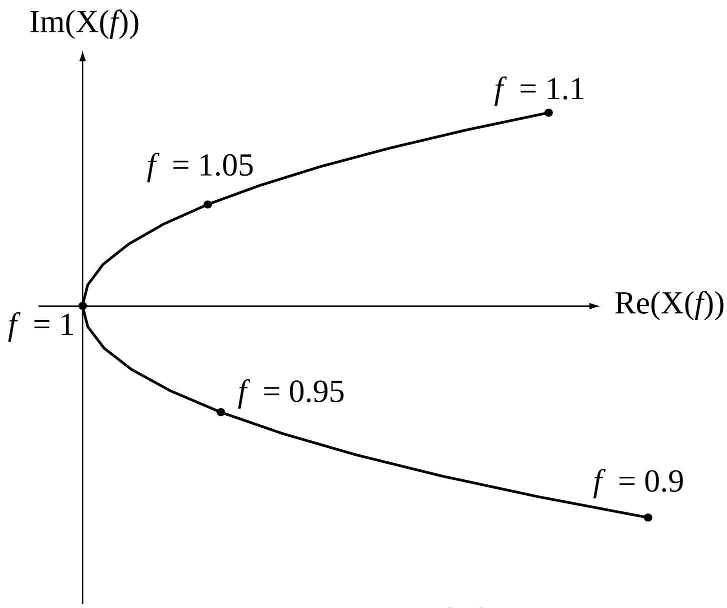


Figure P-17 Plot of the imaginary part of $X(f)$ versus the real part of $X(f)$

At $f = 1$, the plot goes through the origin of the complex plane, tangent to the imaginary axis. Therefore the angle of a vector from the origin to the complex value of $X(f)$ approaches $-\pi / 2$ just before reaching the origin and as it passes through the origin the angle changes suddenly to $+\pi / 2$, agreeing with the plot in Figure P-15. So the phase <u>is discontinuous</u>, even though the complex value of $X(f)$ <u>is continuous</u>! This can only happen where the complex value of $X(f)$ passes through zero. At any other point in the complex plane a phase discontinuity would cause a discontinuity in the complex value of $X(f)$. (<u>Unless</u> the size of the discontinuity of phase is <u>exactly</u> an integer multiple of 2π radians. In the case in which the discontinuity of phase is exactly an integer multiple of 2π radians the phase discontinuity is only apparent, not real, because we can always replace that phase with one which is continuous, making the phase plot again continuous.)

Exercises

(On each exercise, the answers listed are in random order.)

1. Find all the solutions of

(a) $z^2 + 8 = 2$ (b) $z^2 - 2z + 10 = 0$ (c) $7z^2 + 3z + 8 = 5$

Answers: $-0.2143 \pm j0.6186$ $\pm j\sqrt{6}$ $1 \pm j3$

2. If $z_1 = 3 - j6$ and $z_2 = 2 + j8$ and $z = x + jy$, find x and y in each case.

(a) $z = z_1 + z_2$ (b) $z = z_1 - z_2$ (c) $z = z_2 - z_1$

(d) $z = z_1 z_2$ (e) $z = \dfrac{z_1}{z_2}$ (f) $z = \dfrac{z_2}{z_1}$

(g) $z = \dfrac{1}{z_1}$ (h) $z = \dfrac{1}{z_2}$

Answers: $\dfrac{1}{34} - j\dfrac{2}{17}$ $-1 + j14$ $\dfrac{1}{15} + j\dfrac{2}{15}$ $5 + j2$

$54 + j12$ $-\dfrac{14}{15} + j\dfrac{4}{5}$ $1 - j14$ $-\dfrac{21}{34} - j\dfrac{9}{17}$

3. If $z_1 = \dfrac{1 + j2}{5}$ and $z_2 = j(4 - j3)$ and $z = x + jy$, find x and y in each case.

(a) $z = z_1 + z_2$ (b) $z = z_1^* + z_2$ (c) $z = z_2^*$

(d) $z = z_2 + z_2^*$ (e) $z = z_2 - z_2^*$ (f) $z = z_1 z_1^*$

(g) $z = \dfrac{z_1}{z_2^*}$

Answers: $\dfrac{16}{5} + j\dfrac{18}{5}$ $3 - j4$ $\dfrac{1}{5}$ 6 $\dfrac{16}{5} + j\dfrac{22}{5}$ $j8$

$-\dfrac{1}{25} + j\dfrac{2}{25}$

4. If $z_1 = (j-3)^*$ and $z_2 = \dfrac{3-j2}{-4-j}$ find $|z|$ in each case.

(a) $z = z_1$ (b) $z = z_2$ (c) $z = z_1 z_1^*$

(d) $z = z_2 z_2^*$ (e) $z = z_1 + z_2^*$ (f) $z = z_2 z_1^*$

(g) $z = \dfrac{z_1 + z_2}{z_1}$ (h) $z = \dfrac{z_1}{z_1^*}$ (i) $z = \dfrac{z_2}{z_2^*}$

Answers: $\dfrac{\sqrt{4505}}{17}$ 1 $\dfrac{13}{17}$ 10 1 $\sqrt{\dfrac{130}{17}}$ $\sqrt{10}$
$\dfrac{\sqrt{3757}}{17\sqrt{10}}$ $\sqrt{\dfrac{13}{17}}$

5. Find the magnitude and angle of these complex numbers.

(a) $z = 1 + j$ (b) $z = 1 - j$ (c) $z = 3 - j3$

(d) $z = -4 + j3$ (e) $z = (1+j)(-1-j)$ (f) $z = \dfrac{1}{1+j}$

(g) $z = \dfrac{2-j}{1+j3}$ (h) $z = \left(\dfrac{2-j}{1+j3}\right)^*$

Answers: $\sqrt{2}\measuredangle\dfrac{\pi}{4} \pm 2n\pi$ $\dfrac{1}{\sqrt{2}}\measuredangle-\dfrac{\pi}{4}$ $5\measuredangle 2.498$ $\dfrac{1}{\sqrt{2}}\measuredangle 1.713$
$2\measuredangle-\dfrac{\pi}{2}$ $\dfrac{1}{\sqrt{2}}\measuredangle-1.713$ $\sqrt{2}\measuredangle-\dfrac{\pi}{4} \pm 2n\pi$
$3\sqrt{2}\measuredangle-\dfrac{\pi}{4} \pm 2n\pi$

6. Find all the distinct solutions to these equations.

(a) $z^2 = j$ (b) $z^3 = j$ (c) $z^5 = -1$

(d) $z^4 - 3 = j$ (e) $z^3 - 8 = 0$

Answers: $1\measuredangle\frac{\pi}{4}$ or $1\measuredangle-\frac{3\pi}{4}$ $\quad$ $1\measuredangle\frac{\pi}{6}$ or $1\measuredangle\frac{5\pi}{6}$ or $1\measuredangle-\frac{\pi}{2}$

$2\measuredangle 0$ or $2\measuredangle\frac{2\pi}{3}$ or $2\measuredangle-\frac{2\pi}{3}$ $\quad$ $1\measuredangle\frac{\pi}{5}$ or $1\measuredangle\frac{3\pi}{5}$ or -1 or $1\measuredangle-\frac{\pi}{5}$ or $1\measuredangle-\frac{3\pi}{5}$

$1.333\measuredangle 0.0804$ or $1.333\measuredangle 1.651$ or $1.333\measuredangle-3.061$ or $1.333\measuredangle-1.490$

7. Evaluate these exponential functions.

(a) $e^{j\pi}$ (b) $e^{j\pi/2}$ (c) $e^{-j\pi/2}$ (d) $e^{j3\pi/2}$

(e) $e^{j\pi}+e^{-j\pi}$ (f) $e^{j\pi/2}+\left(e^{j\pi/2}\right)^*$ (g) $e^{\pi}+e^{-\pi}$

(h) $e^{\pi/2}+\left(e^{\pi/2}\right)^*$

Answers: 9.621, j, -2, -j, 23.184, 0, -1, -j

8. If $z = x + jy = Ae^{j\theta}$, find x, y, A and θ.

(a) $z=4e^{j\pi/2}$ (b) $z=4e^{1-j\pi/2}$ (c) $z=\left(4e^{1-j\pi/2}\right)\left(j2e^{-1-j\pi/2}\right)$

(d) $z=\left(-10e^{j3\pi/2}\right)^3$ (e) $z=\left(e^{j3\pi/2}\right)^{3/2}$

Answers:

$$x=0\ ,\ y=4\ ,\ A=4\ ,\ \theta=\frac{\pi}{2}+2n\pi$$

$$x=0\ ,\ y=-1000\ ,\ A=1000\ ,\ \theta=-\frac{\pi}{2}+2n\pi$$

$$x=0\ ,\ y=-8\ ,\ A=8\ ,\ \theta=-\frac{\pi}{2}+2n\pi$$

$$x=\frac{1}{\sqrt{2}}\ ,\ y=\frac{1}{\sqrt{2}}\ ,\ A=1\ ,\ \theta=\frac{\pi}{4}+2n\pi$$

or

$$x=-\frac{1}{\sqrt{2}}\ ,\ y=-\frac{1}{\sqrt{2}}\ ,\ A=1\ ,\ \theta=\frac{5\pi}{4}+2n\pi$$

$$x=0\ ,\ y=-10.87\ ,\ A=10.87\ ,\ \theta=-\frac{\pi}{2}+2n\pi$$

9. Using MATLAB plot the magnitude and phase of the following complex functions of the real independent variable t, f or ω over the range indicated.

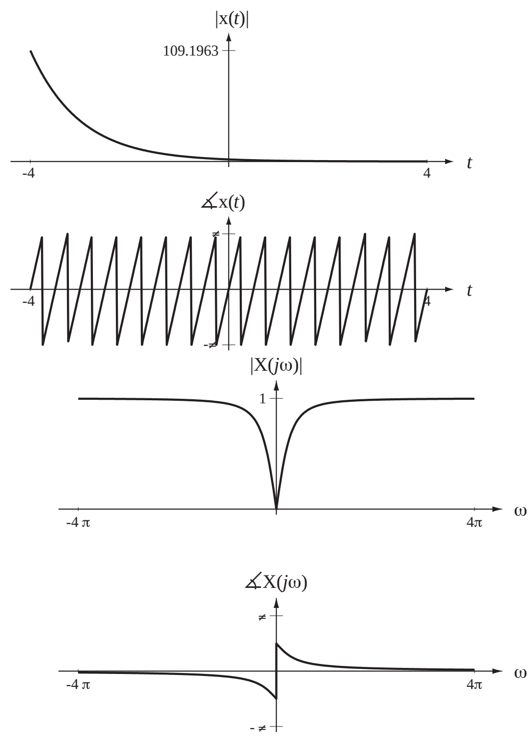
(a) $\mathrm{x}(t) = 2e^{-j4\pi t} \quad , \quad -1 < t < 1$

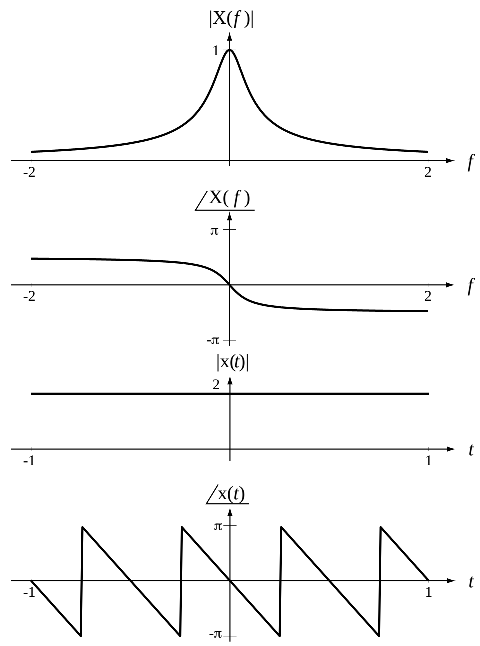
(b) $\mathrm{x}(t) = 2e^{(-1+j4\pi)t} \quad , \quad -4 < t < 4$

(c) $\mathrm{X}(\mathrm{f}) = \dfrac{1}{1 + j2\pi \mathrm{f}} \quad , \quad -2 < \mathrm{f} < 2$

(d) $\mathrm{X}(j\omega) = \dfrac{j\omega}{1 + j\omega} \quad , \quad -4\pi < \omega < 4\pi$

Answers:





10. Convert these complex numbers to the polar form $Ae^{j\theta}$ with an angle in the range, $-\pi < \theta \le \pi$.

(a) $1+j$ (b) $3-j2$ (c) $-j$ (d) $-j+1$

11. Convert these complex numbers to the rectangular form $x + jy$.

(a) $e^{j\pi}$ (b) $4\measuredangle 45°$ (c) $3e^{2-j\frac{\pi}{4}}$ (d) $10e^{-j\frac{11\pi}{4}}$

12. Find the numerical value of z in both the rectangular and polar forms.

(a) $z = 2e^{-1+j\pi/2} + 4 - j2$ (b) $z = (1-j)(4+j5)^2$ (c) $z = \left|(-j)^3\right|$

(d) $z = \dfrac{2e^{j\pi}}{(1+j)^4}$ (e) $z = \dfrac{2e^{-j\pi}}{(1-j)^4}$ (f) $z = e^{1+j} - e^{1-j}$

13. Using MATLAB, plot graphs of the magnitude and phase of the following functions of the real variable f over the range indicated.

(a) $\mathrm{X}(f) = \dfrac{10}{1 + j\dfrac{f}{100}}, -400 < f < 400$

(b) $\mathrm{X}(f) = \dfrac{j10f}{1 + j\dfrac{f}{100}} \, , -400 < f < 400$

(c) $\mathrm{X}(f) = e^{j\pi f} - e^{j2\pi f} \, , -8 < f < 8$

(d) $\mathrm{X}(f) = \dfrac{5}{1 - f^2 + j\dfrac{f}{4}} \, , -4 < f < 4$

(e)

$$\mathrm{X}(j\omega) = \frac{1}{\left(j\omega - e^{j3\pi/4}\right)\left(j\omega - e^{j5\pi/4}\right)\left(j\omega - e^{-j3\pi/4}\right)\left(j\omega - e^{-j5\pi/4}\right)} \; , \; -2\pi < \omega < 2\pi$$

(f)

$$\mathrm{X}(j\omega) = \frac{(j\omega)^4}{\left(j\omega - e^{j3\pi/4}\right)\left(j\omega - e^{j5\pi/4}\right)\left(j\omega - e^{-j3\pi/4}\right)\left(j\omega - e^{-j5\pi/4}\right)} \; , \; -2\pi < \omega < 2\pi$$

14. (a) Show that the magnitude of the complex function e^{jx}, x a real number, is one, regardless of the value of x.

(b) Find the simplest expression you can for the phase of e^{jx} as a function of x.

(c) Graph the phase of e^{jx} by hand and then write a MATLAB program to graph the same phase. The graphs should extend over values of x in the range, $-4\pi < x < 4\pi$. If the graphs are different, explain why.