Appendix L - Differential and Difference Equations

L.1 Introduction

Differential equations are those in which an equality is expressed in terms of a function of one or more independent variables and derivatives of the function with respect to one or more of those independent variables. Difference equations are those in which an equality is expressed in terms of a function of one or more independent variables and finite differences of the function. Differential equations are important in signal and system analysis because they describe the dynamic behavior of continuous-time (CT) physical systems. Difference equations are important in signal and system analysis because they describe the dynamic behavior of discrete-time (DT) systems. Discrete-time is equally-spaced points in time, separated by some time difference Δt. In DT signals and systems the behavior of a signal and the action of a system are known only at discrete points in time and are not defined between those discrete points in time.

Differential equations have several properties by which they are classified, *linear and non-linear, ordinary and partial, homogeneous and inhomogeneous.* They are also classified by their *order*, which is the highest order of a derivative in the equation after it is put into a standard form, and by the coefficients of the derivatives which may either be constants, or functions of the independent variable. Difference equations are classified in a similar manner in which the order of the difference equation is the highest order difference after being put into standard form. Fortunately the great majority of systems are described (at least approximately) by the types of differential or difference equations that are easiest to solve, ordinary, linear differential or difference equations with constant coefficients. This appendix covers only equations of that type.

L.2 Homogeneous Constant-Coefficient Linear Differential Equations

Let us begin with an example of the simplest differential equation, a homogeneous, first-order, linear, ordinary differential equation

$$2\frac{d\,\mathrm{y}(t)}{dt} + 7\,\mathrm{y}(t) = 0. \qquad \text{(L.1)}$$

We can streamline the notation by indicating differentiation by

$$\frac{d\,\mathrm{y}(t)}{dt} = \mathrm{y}' \ , \ \frac{d^2\,\mathrm{y}(t)}{dt^2} = \mathrm{y}'' \ , \ \frac{d^3\,\mathrm{y}(t)}{dt^3} = \mathrm{y}''' \ , \ \text{etc}\ldots \qquad \text{(L.2)}$$

In (L.2) t is the independent variable and y is the dependent variable, a function of t. Rewriting (L.1) in the streamlined notation,

$$2\,y' + 7\,y = 0\,. \tag{L.3}$$

A homogeneous, linear, ordinary differential equation is a linear combination of the dependent variable and its derivatives, set equal to zero. We can rearrange (L.3) into

$$y' = -7\,y/2\,.$$

This equation must be satisfied for any arbitrary value of the independent variable t. That means that y, which is a function of t, must have the same functional form as y'. The only function which has that property is the exponential function because

$$\frac{d}{dt}\left(e^{\lambda t}\right) = \lambda e^{\lambda t}\,.$$

Therefore the functional form of the solution of (L.1) is $y(t) = e^{\lambda t}$, where λ is a constant, as yet undetermined. The exponential function is unique to or characteristic of this type of differential equation because it is the only functional form that can solve it. The characteristic function of a differential equation is commonly referred to as the *eigenfunction* after the German word Eigenfunktion meaning characteristic function. To check the validity of this solution form, we put $y(t) = e^{\lambda t}$ into (L.1) and perform the indicated operations,

$$2\,y' + 7\,y = 2\lambda e^{\lambda t} + 7e^{\lambda t} = 0$$

or $2\lambda + 7 = 0$.

This equation is sometimes referred to as the *characteristic equation* associated with the differential equation. It is an algebraic equation and is satisfied if $\lambda = -7/2$. Then $y(t) = e^{\lambda t}$ is a solution of (L.1) if $\lambda = -7/2$. But it is not the most general solution. We can multiply by an arbitrary constant K to get a solution form $y(t) = Ke^{\lambda t}$. When we put it into (L.3) we get

$$2\,y' + 7\,y = 2K\lambda e^{\lambda t} + 7Ke^{\lambda t} = 0$$

or $2\lambda + 7 = 0$ as before. This is the most general form of solution. The particular value of λ which solves the characteristic equation is called an *eigenvalue* and the solution works for any arbitrary value of K.

An exact value of K can only be specified by using more information than is contained in the differential equation itself. It is found by applying *boundary conditions*. In order to specify K one must know the value of y or its first derivative at some particular value of t. Suppose it is known that when t is 0, y is 2. Then from $y(t) = Ke^{\lambda t}$

$$2 = Ke^{-(7/2)(0)} = K \Rightarrow K = 2\,.$$

We see now that the arbitrary constant K is needed to satisfy both the equation and the boundary condition. Then the full numerical solution of with boundary conditions is $y(t) = 2e^{-7t/2}$ (Figure L-1) .

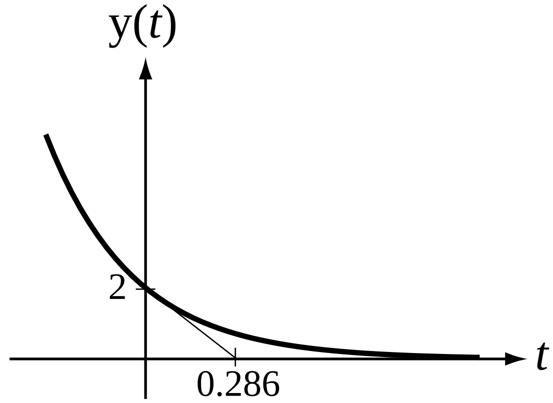


Figure L-1 The complete solution of the first-order, linear, constant-coefficient, ordinary differential equation with boundary conditions

In this case the boundary condition was given at $t = 0$ and, if t represents time, this type of boundary condition is called an *initial condition*. By analogy to the procedures followed in this example, the solution of any equation of the form $y' = ay$ can be found.

Inhomogeneous Constant-Coefficient Linear Differential Equations

The next step up in equation complexity is the *inhomogeneous* first-order, linear, ordinary differential equation. An inhomogeneous, linear, ordinary differential equation is a linear combination of the dependent variable and its derivatives set equal to a function of the independent variable which is often called the *forcing function*. For example,

$$2\,y' + 7\,y = 4\cos(3t) \ . \tag{L.5}$$

We can rewrite (L.5) as

$$y' = -7\,y/2 + 2\cos(3t) \ . \tag{L.6}$$

(This form, in which the first derivative of the function is on the left side of the equation and the right side has the function itself followed by the forcing function, is a standard way of writing differential equations which is often used in systems of multiple differential equations.) We need to find a function $y(t)$ for which (L.6) is satisfied for any arbitrary t. If we choose y to be a cosine function of t, that yields a cosine on the right side but the derivative will be a sine function of t which will appear on the left side and that does not work. We could choose a sine function of t and that would make the derivative have the right form, but not the function itself. But if we choose a linear combination of a sine and a cosine maybe we can arrange to have the two sine functions cancel somehow and leave just a cosine function. Let's try a solution of the form $y(t) = K_1\sin(3t) + K_2\cos(3t)$. Substituting that into

$$2\left[3K_1\cos(3t) - 3K_2\sin(3t)\right] + 7\left[K_1\sin(3t) + K_2\cos(3t)\right] = 4\cos(3t) \ .$$

For this to be a solution for any arbitrary t, since cosines and sines are different functions of t, the cosine parts on each side must be equal and the sine parts on each side must be equal, independently. That is

$$6K_1\cos(3t) + 7K_2\cos(3t) = 4\cos(3t)$$

$$-6K_2\sin(3t) + 7K_1\sin(3t) = 0$$

or

$$6K_1 + 7K_2 = 4 \quad \text{and} \quad -6K_2 + 7K_1 = 0 .$$

Solving,

$$K_1 = 24/85 \text{ and } K_2 = 28/85 .$$

Therefore $y(t) = (24/85)\sin(3t) + (28/85)\cos(3t)$ is one solution of (L.5).

The solution form $y(t) = K_1\sin(3t) + K_2\cos(3t)$ could have been written in an equivalent form by using the trigonometric identity

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) .$$

Then

$$y(t) = K\cos(3t+\theta)$$

where

$$K = \sqrt{K_1^2 + K_2^2} \quad \text{and} \quad \tan(\theta) = -K_1/K_2 .$$

Sometimes this form is more convenient or appropriate.

The method used to find the coefficients K_1 and K_2 is called the *method of undetermined coefficients* and it can be applied to find a solution to any linear ordinary inhomogeneous differential equation with constant coefficients. It is important to point out here that this is one solution to the equation but not the total solution. This solution is called the *particular* solution and will be denoted here by $y_p(t)$. The total solution is the sum of the particular solution and the solution of the homogeneous form of the equation $y_h(t)$ which was found in the previous example. The sum of the two is also a solution of $2y' + 7y = 4\cos(3t)$ because $2y_h' + 7y_h = 0$ and if the total solution is $y = y_h + y_p$ and that is substituted into (L.5) we get

$$2(y_h + y_p)' + 7(y_h + y_p) = 2(y_h' + y_p') + 7(y_h + y_p) = 4\cos(3t)$$

or

$$\underbrace{2y_h' + 7y_h}_{=0} + 2y_p' + 7y_p = 4\cos(3t)$$

which is the same as the original inhomogeneous equation, (L.5). Therefore the total solution to (L.5) is

$$y(t) = Ke^{-7t/2} + \frac{24}{85}\sin(3t) + \frac{28}{85}\cos(3t)$$

where again, K must be found by matching boundary conditions, but this time with the total solution, not just the homogeneous solution. (In the previous example the homogeneous solution was the total solution.) Suppose y is 0 when t is $\pi / 3$. Then

$$0 = Ke^{-7\pi/6} + \frac{24}{85}\sin(\pi) + \frac{28}{85}\cos(\pi) \Rightarrow K = -12.86 .$$

We can always add the homogeneous solution to the particular solution to get a total solution because, when we substitute it into the differential equation and do the differentiations, it always adds to zero. The homogeneous solution is also called the *transient* solution because for stable physical systems described by this kind of equation, the homogeneous solution decays away with time or the *natural* response because its form indicates the nature of the system described by the differential equation. The particular solution is also called the *steady-state* solution because it is the solution which persists after the transient solution has died away or the *forced* response because it is the part of the solution that is forced to exist by the action of the forcing function.

We found the solution above by assuming a particular solution in a form consisting of a linear combination of sines and cosines $\mathrm{y}(t) = K_1 \sin(3t) + K_2 \cos(3t)$. Since sines and cosines can be expressed in terms of complex exponentials we could change this solution form to

$$\mathrm{y}(t) = \frac{K_2 - jK_1}{2} e^{j3t} + \frac{K_2 + jK_1}{2} e^{-j3t}$$

or $\mathrm{y}(t) = K_1' e^{j3t} + K_2' e^{-j3t}$ where

$$K_1' = \frac{K_2 - jK_1}{2} \quad \text{and} \quad K_2' = \frac{K_2 + jK_1}{2} . \tag{L.7}$$

Then, substituting this solution form into the original equation

$$2\,\mathrm{y}' + 7\,\mathrm{y} = 4\cos(3t) = 2\left(e^{j3t} + e^{-j3t}\right)$$

we get

$$j6K_1' e^{j3t} - j6K_2' e^{-j3t} + 7K_1' e^{j3t} + 7K_2' e^{-j3t} = 2\left(e^{j3t} + e^{-j3t}\right).$$

Notice that when we equate like functional forms on both sides we get

$$j6K_1' e^{j3t} + 7K_1' e^{j3t} = 2e^{j3t} \quad \text{and} \quad -j6K_2' e^{-j3t} + 7K_2' e^{-j3t} = 2e^{-j3t}$$

or

$$(7 + j6)K_1' = 2 \quad \text{and} \quad (7 - j6)K_2' = 2 \ . \tag{L.8}$$

For (L.8) to be satisfied $K_1' = \left(K_2'\right)^*$, a requirement we have already seen in (L.7). Solving the left-hand equation in (L.8),

$$K_1' = \frac{14 - j12}{85} .$$

Then

$$K_2' = \frac{14 + j12}{85}$$

and

$$y_p(t) = \frac{14 - j12}{85} e^{j3t} + \frac{14 + j12}{85} e^{-j3t} .$$

This solution form can be converted into

$$y_p(t) = \frac{28\cos(3t) + 24\sin(3t)}{85}$$

which is exactly the same as the previous solution form. Since in this solution method the undetermined coefficients must always occur in complex conjugate pairs, we can abbreviate the solution process by finding the solution to $2\,y' + 7\,y = 4e^{j3t}$ which is

$$y_p(t) = \frac{4}{7 + j6} e^{j3t} = \frac{28 - j24}{85}\left(\cos(3t) + j\sin(3t)\right)$$

or

$$y_p(t) = \frac{28\cos(3t) + 24\sin(3t) - j24\cos(3t) + j28\sin(3t)}{85} .$$

Observe that the real part of this solution is the same as the solution of the original equation.

We found a particular solution for a particular forcing function $4\cos(3t)$. Of course if the numerical coefficients changed we could still find a solution by the same technique. But what happens if the functional form of the forcing function changes? Then the functional form of the solution must also change. There are many possible functional forms of forcing functions but the most common ones that occur in engineering practice are ones which are at least piecewise continuous and differentiable with a finite number of unique functional forms of the derivatives. Commonly-occurring functions which have these properties are $g(t) = A_0 + A_1 t + A_2 t^2 + \cdots + A_N t^N$, $g(t) = Ae^{at}$, $g(t) = A\cos(at)$, $g(t) = A\sin(at)$ and sums of products of these functions, for example,

$$g(t) = Ae^{at}\cos(bt) + Be^{ct}\sin(dt).$$

As long as we restrict ourselves to forcing functions of these forms, we can always find a particular solution by proposing a solution form containing the forcing function form and all its unique derivatives. This may sound unnecessarily restrictive. After all, do all real systems have forcing functions of these few forms? No. But, as it turns out, all forcing functions with any engineering usefulness can be expressed as linear combinations of functions of these forms. In fact linear combinations of complex sinusoids are sufficient to describe any forcing function with engineering usefulness.

The next step up in complexity is to a higher-order differential equation. For example

$$y'' + 5y' + 3y = 6t^2 . \tag{L.9}$$

The total solution is the sum of the homogeneous and particular solutions. By reasoning similar to that above for the first-order, linear, constant-coefficient, differential equation, any solution of the homogeneous equation

$$y'' + 5y' + 3y = 0 \tag{L.10}$$

must be of the functional form $e^{\lambda t}$. Substituting into (L.10) and solving,

$$\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 3e^{\lambda t} = 0$$

or

$$\lambda^2 + 5\lambda + 3 = 0 . \tag{L.11}$$

This characteristic equation (L.11) is quadratic and there are two solutions $\lambda_1 = -0.6972$ and $\lambda_2 = -4.303$. So for this second-order differential equation there are two eigenvalues. Which one should we choose? Can we use both? We could propose the solution, $y_{1h}(t) = K_{1h} e^{\lambda_1 t}$ as we did in solving the first-order equation. This solution satisfies the equation but so does the solution $y_{2h}(t) = K_{2h} e^{\lambda_2 t}$. We would like to find the most general solution possible. If we put the sum of these two solutions into the homogeneous differential equation we get

$$\lambda_1^2 K_{1h} e^{\lambda_1 t} + 5\lambda_1 K_{1h} e^{\lambda_1 t} + 3K_{1h} e^{\lambda_1 t} + \lambda_2^2 K_{2h} e^{\lambda_2 t} + 5\lambda_2 K_{2h} e^{\lambda_2 t} + 3K_{2h} e^{\lambda_2 t} = 0$$

Then, substituting in the eigenvalues,

$$\underbrace{\left[(-0.6972)^2 + 5(-0.6972) + 3\right]}_{=0} K_{1h} e^{\lambda_1 t} + \underbrace{\left[(-4.303)^2 + 5(-4.303) + 3\right]}_{=0} K_{2h} e^{\lambda_2 t} = 0$$

and the homogeneous equation is also satisfied by the sum of these two solutions,

$$y_h(t) = K_1 e^{-0.6972t} + K_2 e^{-4.303t}$$

which is the most general possible solution of the homogeneous equation. This result can be generalized to a differential equation of any order. The solution of the homogeneous equation is a linear combination of the eigenfunctions, one for each unique eigenvalue.

The only fly in the ointment occurs when any two eigenvalues are the same. Then the corresponding two eigenfunctions are not independent and can be combined into one eigenfunction. This happens only rarely in practice so it is not of great practical significance. In such a case the needed extra eigenfunction has the functional form, $te^{\lambda t}$.

A particular solution of (L.9) can be found by proposing a solution which is a linear combination of the forcing function and all its unique derivatives of the form,

$y_p(t) = At^2 + Bt + C$. Substituting into (L.9) $2A + 5(2At + B) + 3(At^2 + Bt + C) = 6t^2$ and solving, $A = 2$, $B = -20/3$, $C = 88/9$. Therefore the total solution is

$$y(t) = y_h(t) + y_p(t) = K_1 e^{-0.6972t} + K_2 e^{-4.303t} + 2t^2 - \frac{20}{3}t + \frac{88}{9} \tag{L.12}$$

and the two remaining constants K_1 and K_2 must be found be applying two independent boundary conditions. (The number of boundary conditions needed is always equal to the order of the differential equation.)

Systems of Linear Differential Equations

So far we have only considered the solution of a single differential equation for a single unknown function. A very common situation in signal and system analysis is the solution of systems of differential equations. Most systems of interest are described by more than one differential equation. An example that will introduce some solution methods is the relatively simple two-differential-equation system,

$$\begin{aligned} y_1' + 5y_1 + 2y_2 &= 10 \\ y_2' + 3y_2 + y_1 &= 0 \end{aligned} \tag{L.13}$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 0$.

We will solve this system of equations using two different methods. The first method will be an ad-hoc method in which we combine the two first-order equations, each in two functions, into two second-order equations, each in only one function, and then solve these equations by the methods of the previous section. The second method is a more systematic technique of solving the two original first-order equations simultaneously. The second method is a natural stepping stone to techniques for solving systems of arbitrary numbers of first-order equations using matrix methods, which will be introduced later.

We can rearrange the second equation in (L.13) to form

$$y_1 = -y_2' - 3y_2 \ . \tag{L.14}$$

Then, substituting (L.14) into the first equation in (L.13) we get

$$-y_2'' - 3y_2' + 5(-y_2' - 3y_2) + 2y_2 = 10$$

or

$$y_2'' + 8y_2' + 13y_2 = -10 \, .$$

We can solve this equation using the techniques of the previous section and, when we do we get the characteristic equation $\lambda^2 + 8\lambda + 13 = 0$, the eigenvalues, $\lambda_1 = -4 + \sqrt{3}$ and $\lambda_2 = -4 - \sqrt{3}$, the particular solution $y_p = -10/13$, and the total solution

$$y_2 = K_1 e^{-\lambda_1 t} + K_2 e^{-\lambda_2 t} - 10/13. \tag{L.15}$$

We can apply the two initial conditions. The condition $y_2(0) = 0$ applied to (L.15) yields $K_1 + K_2 = 10/13$ and we can use (L.14) with (L.15) and the initial condition $y_1(0) = 1$ to form $(\lambda_1 + 3)K_1 + (\lambda_2 + 3)K_2 = 17/13$. Solving, we get $K_1 = 0.9841$ and $K_2 = -0.2151$ and a total numerical solution

$$y_2(t) = 0.9841 e^{-2.268t} - 0.2151 e^{-5.732t} - 0.769. \tag{L.16}$$

Now we can use (L.14) along with (L.16) to find $y_1(t)$,

$$y_1(t) = -0.720 e^{-2.268t} - 0.588 e^{-5.732t} + 2.308 . \tag{L.17}$$

Notice that because of the relationship between the two solution functions $y_1(t)$ and $y_2(t)$ in (L.13), that the eigenvalues for the two functions are the same. Only the constants multiplying the eigenfunctions and the particular solutions are different. The fact that the eigenvalues are the same is a result of the fact that the two first-order differential equations are coupled, therefore not independent.

Now we will solve the system, (L.13), using a different technique. Since each equation in (L.13) is first-order we assume the solution forms $y_{1h}(t) = K_{1h} e^{\lambda t}$ and $y_{2h}(t) = K_{2h} e^{\lambda t}$. Then, substituting these forms into (L.13) and simplifying we get two, first-order, coupled characteristic equations

$$\begin{aligned} (\lambda + 5)K_{1h} + 2K_{2h} &= 0 \\ K_{1h} + (\lambda + 3)K_{2h} &= 0 \end{aligned} . \tag{L.18}$$

This is a system of two equations in three unknowns so we should not expect to be able to find a unique solution, but in this case, because of the form of the equations, _we can solve for λ_. Rearranging (L.18)

$$\begin{aligned} \frac{K_{1h}}{K_{2h}} &= -\frac{2}{\lambda + 5} \\ \frac{K_{1h}}{K_{2h}} &= -(\lambda + 3) \end{aligned} \tag{L.19}$$

Then, equating the two equations in (L.19) ,

$$\frac{2}{\lambda + 5} = \lambda + 3$$

or $\lambda^2 + 8\lambda + 13 = 0$. This is exactly the same characteristic equation we got in the previous solution and the eigenvalues are again

$$\lambda_1 = -4 + \sqrt{3} \text{ and } \lambda_2 = -4 - \sqrt{3} . \tag{L.20}$$

So there are two values of λ for which (L.18) can be satisfied. This is a little different from the ordinary experience of solving two simultaneous equations. Even though we have three unknowns we are still able to solve for one of them. But, when we do, we get two possible values for that unknown instead of one. We have not found unique values but we have narrowed the field of possible values.

Now let's do what one would ordinarily do in solving algebraic equations, choose one eigenvalue λ_1 and put it into the equations in (L.18) and try to find the arbitrary constants K_{1h} and K_{2h} (with numerical values substituted into (L.18))

$$K_{1h} = -\frac{2}{\sqrt{3}+1} K_{2h}$$

and

$$\left[-\frac{2}{\sqrt{3}+1} - 1 + \sqrt{3}\right] K_{2h} = 0 . \tag{L.21}$$

Equation (L.21) can be satisfied if K_{2h} is zero or the coefficient of K_{2h} equals zero, or both. In other words, to find a non-trivial solution ($K_{2h} \neq 0$) the coefficient $\left[-\frac{2}{\sqrt{3}+1} - 1 + \sqrt{3}\right]$ must be zero. Simplifying (L.21) we get $(0)K_{2h} = 0$. Therefore, for this eigenvalue, we know it is possible for K_{2h} to be non-zero but we don't yet know what it is. This should have been expected because we are trying to solve a system of two equations in three unknowns and there is no unique solution. If we do the same thing with the other eigenvalue we get the same result. But we can say one more thing that is useful about the arbitrary constants K_{1h} and K_{2h}. From (L.19)

$$\frac{K_{1h}}{K_{2h}} = -(\lambda + 3) .$$

So, for the first eigenvalue,

$$\frac{K_{1h}}{K_{2h}} = 1 - \sqrt{3} = -0.732 .$$

Similarly, for the second eigenvalue,

$$\frac{K_{1h}}{K_{2h}} = 1 + \sqrt{3} = 2.732$$

We have not yet found unique solutions for all three unknowns but we have established certain relationships among them. To accommodate these results for the two eigenvalues we need homogeneous solutions which are linear combinations of the two eigenfunctions corresponding to the two eigenvalues. So we assume solutions of the forms,

$$\mathrm{y}_{1h}(t) = K_{11h} e^{\lambda_1 t} + K_{12h} e^{\lambda_2 t} \text{ and } \mathrm{y}_{2h}(t) = K_{21h} e^{\lambda_1 t} + K_{22h} e^{\lambda_2 t}$$

where

$$\frac{K_{11h}}{K_{21h}} = -0.732 \text{ and } \qquad \frac{K_{12h}}{K_{22h}} = 2.732 \tag{L.22}$$

We need something to establish the exact values of the constants instead of just their ratios and that requires using the initial conditions. But before applying the initial conditions we need a particular solution to complete the total solution. Since the forcing function is a constant we can assume particular solutions

$$\mathrm{y}_{1p}(t) = K_{1p} \quad \text{and} \quad \mathrm{y}_{2p}(t) = K_{2p} \tag{L.23}$$

Doing that and solving, we get

$$K_{1p} = 30/13 \ , \ K_{2p} = -10/13. \tag{L.24}$$

So the total solutions are of the forms

$$\mathrm{y}_1(t) = K_{11\mathrm{h}}e^{\lambda_1 t} + K_{12\mathrm{h}}e^{\lambda_2 t} + 30/13 \text{ and } \mathrm{y}_2(t) = K_{21\mathrm{h}}e^{\lambda_1 t} + K_{22\mathrm{h}}e^{\lambda_2 t} - 10/13 \tag{L.25}$$

Since we know two relations (L.22) among the four arbitrary constants there are actually only two unknowns, so we need only two initial conditions. We can now use $\mathrm{y}_1(0) = 1$ and $\mathrm{y}_2(0) = 0$. Applying the initial conditions we get

$$\mathrm{y}_1(0) = K_{11\mathrm{h}} + K_{12\mathrm{h}} + 30/13 = 1 \text{ and } \mathrm{y}_2(0) = K_{21\mathrm{h}} + K_{22\mathrm{h}} - 10/13 = 0. \tag{L.26}$$

These two equations, together with (L.25) lead to the final numerical solution,

$$\mathrm{y}_1(t) = -0.720e^{-2.268t} - 0.588e^{-5.732t} + 2.308 \tag{L.27}$$

and

$$\mathrm{y}_2(t) = 0.9841e^{-2.268t} - 0.2151e^{-5.732t} - 0.769 \tag{L.28}$$

This was only the next step up in differential-equation complexity, a two-differential-equation system, and a simple one at that! Try to imagine what it would be like to solve more complicated systems of differential equations. Fortunately, systematic techniques have been developed to solve systems of equations like this. Although these techniques don't actually reduce the total amount of calculation they do arrange the computations in a way that makes them easy to program on a computer and therefore makes the solutions much easier in practice for humans to obtain.

MATLAB can solve systems of differential equations. Below is the MATLAB `help` message for the command, `DSOLVE`.

```
DSOLVE Symbolic solution of ordinary differential equations.
    DSOLVE('eqn1','eqn2', ...) accepts symbolic equations representing
    ordinary differential equations and initial conditions.  Several
    equations or initial conditions may be grouped together, separated
    by commas, in a single input argument.
```

```
By default, the independent variable is 't'. The independent
Variable may be changed from 't' to some other symbolic variable by
Including that variable as the last input argument.

The letter 'D' denotes differentiation with respect to the
independent variable, i.e. usually d/dt.  A "D" followed by a digit
Denotes repeated differentiation; e.g., D2 is d^2/dt^2.  Any
characters immediately following these differentiation operators are
taken to be the dependent variables; e.g., D3y denotes the third
derivative of y(t). Note that the names of symbolic variables should
not contain the letter "D".

Initial conditions are specified by equations like 'y(a) = b' or
'Dy(a) = b' where y is one of the dependent variables and a and b
are constants.  If the number of initial conditions given is less
than the number of dependent variables, the resulting solutions will
obtain arbitrary constants, C1, C2, etc.

Examples:

   dsolve('Dx = -a*x') returns

     ans = exp(-a*t)*C1

   x = dsolve('Dx = -a*x','x(0) = 1','s') returns

     x = exp(-a*s)

   y = dsolve('(Dy)^2 + y^2 = 1','y(0) = 0') returns

     y =
     [  sin(t)]
     [ -sin(t)]

   S = dsolve('Df = f + g','Dg = -f + g','f(0) = 1','g(0) = 2')
   returns a structure S with fields

     S.f = exp(t)*cos(t)+2*exp(t)*sin(t)
     S.g = -exp(t)*sin(t)+2*exp(t)*cos(t)

   dsolve('Df = f + sin(t)', 'f(pi/2) = 0')
   dsolve('D2y = -a^2*y', 'y(0) = 1, Dy(pi/a) = 0')
   S = dsolve('Dx = y', 'Dy = -x', 'x(0)=0', 'y(0)=1')
   S = dsolve('Du=v, Dv=w, Dw=-u','u(0)=0, v(0)=0, w(0)=1')
   w = dsolve('D3w = -w','w(0)=1, Dw(0)=0, D2w(0)=0')

See also SOLVE, SUBS.
```

L.3 Linear Ordinary Difference Equations

Finite Difference Approximations to a Derivative

To illustrate a connection between difference equations and differential equations, let us begin with (L.1), the homogeneous, first order, constant-coefficient ordinary differential equation in the previous section,

$$2\frac{d\,\mathrm{y}(t)}{dx} + 7\,\mathrm{y}(t) = 0 \ , \tag{L.29}$$

and approximate it by a difference equation. We can do this by approximating derivatives by finite differences. Recall these definitions of a derivative,

$$\frac{d\,\mathrm{y}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\mathrm{y}(t+\Delta t) - \mathrm{y}(t)}{\Delta t} \ ,$$

$$\frac{d\,\mathrm{y}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\mathrm{y}(t) - \mathrm{y}(t-\Delta t)}{\Delta t}$$

and

$$\frac{d\,\mathrm{y}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\mathrm{y}(t+\Delta t) - \mathrm{y}(t-\Delta t)}{2\Delta t} \ .$$

At any point at which $\mathrm{y}(t)$ is differentiable, any of these definitions of a derivative yield exactly the same result when the limit is taken. A derivative in continuous time can be approximated by a finite difference in discrete time by

$$\frac{\mathrm{y}((n+1)\Delta t) - \mathrm{y}(n\Delta t)}{\Delta t} \ .$$

This is called a forward difference because it uses the present or current value of y $\mathrm{y}(n\Delta t)$ and the next or future value of y $\mathrm{y}((n+1)\Delta t)$. Similarly

$$\frac{\mathrm{y}(n\Delta t) - \mathrm{y}((n-1)\Delta t)}{\Delta t}$$

is a backward difference and

$$\frac{\mathrm{y}((n+1)\Delta t) - \mathrm{y}((n-1)\Delta t)}{2\Delta t}$$

is a central difference. In the limit as Δt approaches zero these are all the same, but in discrete time, Δt is fixed and is not zero and these three approximations to a continuous-time derivative are, in general, different.

As an illustration we will convert the differential equation, (L.29), to a difference equation by using a forward-difference approximation,

$$2\frac{\mathrm{y}((n+1)\Delta t) - \mathrm{y}(n\Delta t)}{\Delta t} + 7\,\mathrm{y}(n\Delta t) = 0 \ . \tag{L.30}$$

To simplify the notation let $y[n] = y(n\Delta t)$ where the square brackets, $[\cdot]$, distinguish a function of discrete time from a function of continuous time which is indicated by using parentheses, $(\cdot)$. In this notation, time is not explicitly indicated but, since the time between consecutive discrete-time values of the function, y is always Δt, we do not need to explicitly indicate time. Using the simplified notation, (L.30) becomes

$$2\,\frac{y[n+1] - y[n]}{\Delta t} + 7\,y[n] = 0$$

or

$$2\left(y[n+1] - y[n]\right) + 7\Delta t\,y[n] = 0 \tag{L.31}$$

which is a homogeneous difference equation.

Some authors use the notation, $y_n = y(n\Delta t)$ for the *n*th value of y. This is an exact equivalent of $y[n]$.

First differences are analogous to first derivatives. In finite-difference mathematics we can denote a first difference of a discrete-time function $x[n]$ by the use of the operator, $\Delta(\cdot)$ and define that operation by $\Delta(x[n]) = x[n+1] - x[n]$. This is the first forward difference of $x[n]$. Then, consistent with that definition, a first backward difference of $x[n]$ would be the first forward difference of $x[n-1]$ or $\Delta(x[n-1]) = x[n] - x[n-1]$. These operations are called *differencing* and are analogous to the operation of differentiation for continuous-time functions. Where it is convenient and unambiguous we can use a shorthand notation for a difference just like the shorthand notation for a derivative of a continuous-time function $\Delta(x[n]) = x'[n]$.

There is a set of rules for differencing which exactly parallels the analogous rules for differentiation (Table 2.3.1.1).

Table 2.3.1.1 Rules for differences and differentiation

Difference	Derivative
Constant, C	
$\Delta(C) = 0$	$\dfrac{dC}{dt} = 0$
Constant times a function	
$\Delta(C\,\mathrm{x}[n]) = C\,\mathrm{x}'[n]$	$\dfrac{d}{dt}(C\,\mathrm{x}(t)) = C\,\mathrm{x}'(t)$
Sum of functions	
$\Delta(\mathrm{x}[n] + \mathrm{y}[n]) = \mathrm{x}'[n] + \mathrm{y}'[n]$	$\dfrac{d}{dt}(\mathrm{x}(t) + \mathrm{y}(t)) = \mathrm{x}'(t) + \mathrm{y}'(t)$
Product of functions	
$\Delta(\mathrm{x}[n]\,\mathrm{y}[n]) = \mathrm{x}[n]\,\mathrm{y}'[n] + \mathrm{y}[n+1]\,\mathrm{x}'[n]$ (Notice the $[n+1]$).	$\dfrac{d}{dt}(\mathrm{x}(t)\,\mathrm{y}(t)) = \mathrm{x}(t)\,\mathrm{y}'(t) + \mathrm{x}'(t)\,\mathrm{y}(t)$
Quotient of functions	
$\Delta\left(\dfrac{\mathrm{x}[n]}{\mathrm{y}[n]}\right) = \dfrac{\mathrm{y}[n]\,\mathrm{x}'[n] - \mathrm{x}[n]\,\mathrm{y}'[n]}{\mathrm{y}[n+1]\,\mathrm{y}[n]}$ (Notice the $[n+1]$).	$\dfrac{d}{dt}\left(\dfrac{\mathrm{x}(t)}{\mathrm{y}(t)}\right) = \dfrac{\mathrm{y}(t)\,\mathrm{x}'(t) - \mathrm{x}(t)\,\mathrm{y}'(t)}{\mathrm{y}^2(t)}$
Power function	
$\Delta(C^n) = C^n(C-1)$	$\dfrac{d}{dt}(t^n) = nt^{n-1}$
Cosine	
$\Delta(\cos(n)) = -2\sin\left(\dfrac{1}{2}\right)\sin\left(n + \dfrac{1}{2}\right)$	$\dfrac{d}{dt}(\cos(t)) = -\sin(t)$
Sine	
$\Delta(\sin(n)) = 2\sin\left(\dfrac{1}{2}\right)\cos\left(n + \dfrac{1}{2}\right)$	$\dfrac{d}{dt}(\sin(t)) = \cos(t)$

In case you are asking why the exponential function was left out of the table, the answer is, it wasn't. It is hiding inside the power function. In the formula $\Delta(C^n) = C^n(C-1)$, C is a constant, possibly complex. Therefore it could be represented by $C = e^{\beta}$ where β is an appropriately-chosen constant, also possibly complex. Then the power function difference becomes

$$\Delta\left(\left(e^{\beta}\right)^n\right) = \Delta\left(e^{\beta n}\right) = e^{\beta n}\left(e^{\beta} - 1\right).$$

Equation, (L.31), is a finite-difference approximation to equation, (L.29). Written with the new shorthand notation (L.31) becomes

$$2\,y'[n] + 7\Delta t\, y[n] = 0 \tag{L.32}$$

Notice the similarity of (L.32) to the first-order differential equation it approximates $2\,y'(t) + 7\,y(t) = 0$. To solve (L.32) we can re-write it in recursion form

$$y[n+1] = \frac{2 - 7\Delta t}{2}\, y[n]. \tag{L.33}$$

In recursion form, the difference equation expresses the next value of y $y[n+1]$ in terms of the present value of y $y[n]$. In words, the $[n+1]$th value of y is a multiple of the *n*th value of y, for any *n*. Equation (L.33) can be re-arranged to form,

$$y[n] = \frac{2}{2 - 7\Delta t}\, y[n+1] \tag{L.34}$$

which expresses the present value of y in terms of the next value of y. Also, from (L.33) we can write

$$y[n] = \frac{2 - 7\Delta t}{2}\, y[n-1] \quad . \tag{L.35}$$

in which the present value is written in terms of the immediate past value. Therefore, by using (L.34) and (L.35) if we know any particular value in the sequence of y[*n*] values we can find all the rest and we can express the entire sequence in terms of any single value in the sequence. For example, if we know y[0],

$$\begin{gathered} \vdots \\ y[-1] = \left(\frac{2 - 7\Delta t}{2}\right)^{-1} y[0] \\ y[0] = \left(\frac{2 - 7\Delta t}{2}\right)^{0} y[0] \\ y[1] = \left(\frac{2 - 7\Delta t}{2}\right)^{1} y[0] \\ y[2] = \left(\frac{2 - 7\Delta t}{2}\right)^{2} y[0] \\ \vdots \end{gathered}$$

or, more compactly,

$$y[n] = y[0]\left(\frac{2 - 7\Delta t}{2}\right)^{n}$$

implying that we need the value of y[0] to exactly determine the sequence values. Of course, any particular value of y is sufficient, so a more general form of the solution is

$$y[n] = K\left(\frac{2 - 7\Delta t}{2}\right)^{n}.$$

Suppose we know that $y[0] = 2$ as in the differential equation example. Then the exact solution is determined just as it was in the case of the solution of differential equations.

The nature of the solution of the difference equation and how closely it approximates the exact solution of the differential equation depends on the choice of Δt. For a good approximation, Δt, should be small compared to the time constant of the system which, in this case, is $2/7$ second. Figure L-2 illustrates the effect of different choices of Δt by graphing the solution of the differential equation and the solution of the difference equation approximation to it for four different choices of Δt.

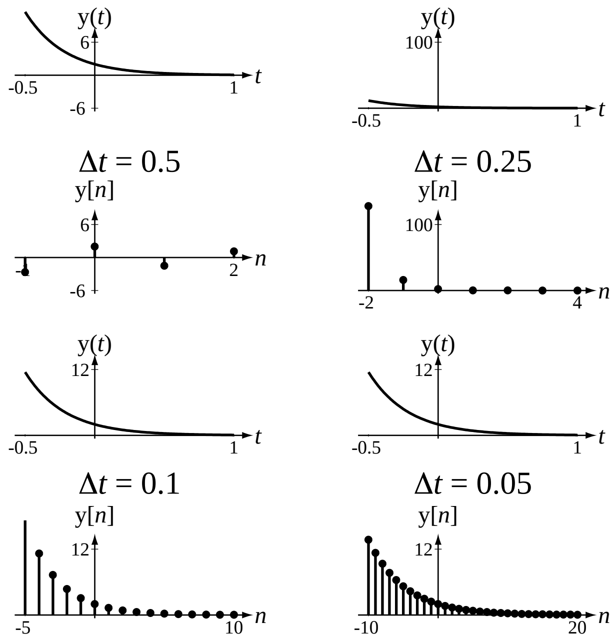


Figure L-2 Solution of the first-order, linear, constant coefficient, ordinary difference equation approximating a differential equation for four different choices of Δt

Notice that when $\Delta t = 0.5$, $y[n]$ is not only inaccurate, it is totally different in character from $y(t)$. It alternates sign with n because the quantity $(2 - 7\Delta t)/2$ is negative. At a value of $\Delta t = 2/7$, which is the time constant of the solution of the original differential equation, the quantity $(2 - 7\Delta t)/2$ is zero. For Δt less than $2/7$ the sign of $y[n]$ no longer alternates with n but if the quantity $(2 - 7\Delta t)/2$ is near zero the solution $y[n]$ is still very inaccurate. As Δt is made smaller the solution $y[n]$ approaches samples of the exact solution $y(t)$ as can be seen in the bottom two plots.

MATLAB has a function for finding differences `diff`. It operates on a vector of length N and returns a vector of forward differences of length $N - 1$. For example,

```
»diff([4 1 -9 3 -4 8])

ans =
```

```
-3   -10    12    -7    12
```

The MATLAB code,

```
dt = 1/16 ; N = 16 ; n = 0:N ; x = sin(2*pi*n*dt) ;
subplot(2,1,1) ;
p = stem(n,x,'k','filled') ;
set(p,'LineWidth',2,'MarkerSize',4) ;
xlabel('n') ; ylabel('x[n]') ; axis([0 16,-1,1]) ;
subplot(2,1,2) ; nd = 0:N-1 ;
p = stem(nd,diff(x),'k','filled') ;
set(p,'LineWidth',2,'MarkerSize',4) ;
xlabel('n') ; ylabel('\Delta(x[n])') ; axis([0 16,-1,1]) ;
```

produces the graphs in Figure L-3,

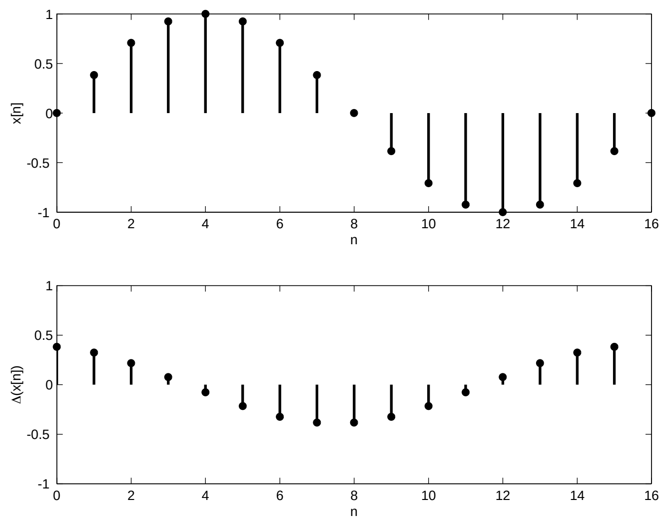


Figure L-3 A DT function and its first forward difference

Homogeneous Linear Constant-Coefficient Difference Equations

The previous example was introduced as a way of approximating the solution of a differential equation. Methods like this are used in numerical analysis for exactly that purpose. However it is important to point out that the solution of difference equations is more than just a way of approximating the solution of differential equations. There are systems which are inherently discrete-time and are not described by differential equations. In those situations the solution of the difference equation is the exact solution because the system is inherently discrete-time.

One classical example of an inherently discrete-time system is a financial system in which interest on an investment is accrued at discrete times. Suppose there is an initial investment of P dollars (the principal) and it earns interest at an annual percentage interest rate r compounded annually. Then Δt is one year and n is the number of the present year. Let the beginning of year zero be the time at which the money was invested and let $A[n]$ be the amount of money in the account at the beginning of the nth year. Then the difference equation describing this DT system is

$$\mathrm{A}[n] = \left(1 + \frac{r}{100}\right)\mathrm{A}[n-1],$$

or

$$\mathrm{A}[n] - \left(1 + \frac{r}{100}\right)\mathrm{A}[n-1] = 0 \qquad \text{(L.36)}$$

and the initial condition is $\mathrm{A}[0] = P$. This can also be written in the form

$$\mathrm{A}'[n-1] - \frac{r}{100}\mathrm{A}[n-1] = 0,$$

which emphasizes its similarity to a first order, homogeneous differential equation. This is a linear, constant-coefficient, homogeneous difference equation. The solution of (L.36) is of the form

$$\mathrm{A}[n] = K\left(1 + \frac{r}{100}\right)^n$$

and the constant, K, is obviously P in this case. Therefore the exact solution of (L.36) is

$$\mathrm{A}[n] = P\left(1 + \frac{r}{100}\right)^n.$$

We have just solved a particular linear, constant-coefficient, homogeneous difference equation. We can generalize to any linear, constant-coefficient homogeneous difference equation. Just as the exponential function of the form $Ae^{\lambda t}$ where A and λ are constants (possibly complex), is the eigenfunction for linear, constant-coefficient homogeneous differential equations of the form

$$\mathrm{y}^{(n)}(t) + a_{n-1}\,\mathrm{y}^{(n-1)}(t) + \cdots + a_2\,\mathrm{y}''(t) + a_1\,\mathrm{y}'(t) + a_0\,\mathrm{y}(t) = 0 \qquad \text{(L.37)}$$

the eigenfunction for linear, constant-coefficient homogeneous difference equations of the form

$$\mathrm{y}[n+k] + a_{n+k-1}\,\mathrm{y}[n+k-1] + \cdots + a_2\,\mathrm{y}[n+2] + a_1\,\mathrm{y}[n+1] + a_0\,\mathrm{y}[n] = 0 \qquad \text{(L.38)}$$

is a function of the form $\mathrm{y}[n] = A\alpha^n$ where A and α are constants (possibly complex). To illustrate a similarity to the eigenfunctions of differential equations, we can express α as an exponential $\alpha = e^{\beta}$ where $\beta = \ln(\alpha)$. Then the form of the eigenfunction is

$$\mathrm{y}[n] = A\alpha^n = A\left(e^{\beta}\right)^n = Ae^{\beta n}$$

which is very similar to the form of the exponential eigenfunctions for differential equations. The difference is that t can have any real value and n can only have real integer values.

Inhomogeneous Linear Constant-Coefficient Difference Equations

Most investment is not done as described in the example in the previous section. It is much more common for an investor to begin an investment program with an initial investment and to add to the account with regular contributions. Let's now modify the previous example to include this effect. Let the yearly contribution, made at the end of each year, be *C*. The new difference equation is

$$\mathrm{A}[n] = \left(1 + \frac{r}{100}\right)\mathrm{A}[n-1] + C.$$

This can be rewritten as

$$\mathrm{A}[n] - \left(1 + \frac{r}{100}\right)\mathrm{A}[n-1] = C \qquad \text{(L.39)}$$

or

$$\mathrm{A}'[n-1] - \frac{r}{100}\mathrm{A}[n-1] = C.$$

Equation (L.39) is an inhomogeneous linear, constant-coefficient, difference equation. We already know the solution

$$\mathrm{A}_h[n] = K_h\left(1 + \frac{r}{100}\right)^n$$

of the corresponding homogeneous difference equation, (L.36), which is denoted here with a subscript, *h* to distinguish it from the particular solution. Now we need the particular solution of the difference equation. Since the forcing function is a constant *C* the particular solution should be in the form of the forcing function and all its unique differences. Of course the first difference (and all higher differences) of a constant is zero so the particular solution is in the form of a constant $\mathrm{A}_p[n] = K_p$. Substituting into (L.39) we get

$$K_p - \left(1 + \frac{r}{100}\right)K_p = C.$$

Solving for K_p, $K_p = -100C/r$. Then the total solution of (L.39) is

$$\mathrm{A}[n] = K_h\left(1 + \frac{r}{100}\right)^n - 100\frac{C}{r}.$$

Applying the initial condition,

$$\mathrm{A}[0] = P = K_h - 100\frac{C}{r} \Rightarrow K_h = P + 100\frac{C}{r}$$

and

$$\mathrm{A}[n] = \left(P + 100\frac{C}{r}\right)\left(1 + \frac{r}{100}\right)^n - 100\frac{C}{r}.$$

To make the example concrete let the parameter values be $P = \$10{,}000$, $C = \$1{,}000$ and $r = 6\%$. Figure L-4 shows the accumulation of account value over a 40 year period.

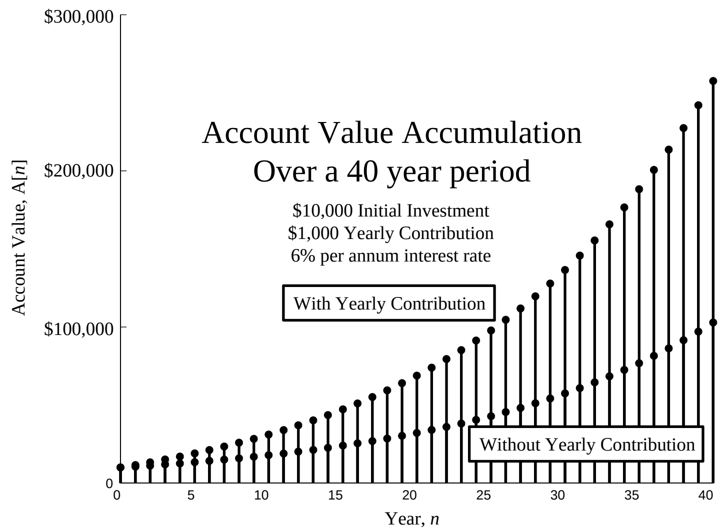


Figure L-4 Account value accumulation over a 40 year period with $10,000 initial investment at 6% interest compounded annually, with and without $1,000 per year contribution

Albert Einstein was once asked by a reporter what the most powerful force in the universe was. Looking at the graph of accumulation of wealth over time in Figure L-4 one can well understand why his answer was "compound interest".

As with differential equations, different forcing functions produce different particular solution forms. Some commonly occurring forcing functions and particular solution forms are (*A*, *K*, *a* and *k* are constants),

Forcing Function	Particular Solution Form
Constant, A	Constant, K
Aa^n	Ka^n
An^k	$K_0 + K_1 n + K_2 n^2 + \cdots + K_k n^k$
$\sin(An)$ or $\cos(An)$	$K_1 \sin(An) + K_2 \cos(An)$
$n^k a^n$	$a^n \left(K_0 + K_1 n + K_2 n^2 + \cdots + K_k n^k\right)$
$a^n \sin(An)$ or $a^n \cos(An)$	$a^n \left[K_1 \sin(An) + K_2 \cos(An)\right]$

Systems of Linear Difference Equations

As was true of differential equation, a very common situation in signal and system analysis is the solution of systems of difference equations. Consider the two-difference-equation system

$$\begin{aligned} 3\,\mathrm{y}_1[n] + 2\,\mathrm{y}_1[n-1] + \mathrm{y}_2[n] &= 0 \\ 4\,\mathrm{y}_2[n] + 2\,\mathrm{y}_2[n-1] + \mathrm{y}_1[n] &= 5 \end{aligned} \tag{L.40}$$

with initial conditions $\mathrm{y}_1[0] = 0$ and $\mathrm{y}_2[0] = 2$. The functional form of the solutions is the eigenfunction α^n. The homogeneous solutions are of the forms $\mathrm{y}_1[n] = K_{h1}\alpha^n$ and $\mathrm{y}_2[n] = K_{h2}\alpha^n$. Substituting the eigenfunction form into (L.40) and simplifying we get the characteristic equations,

$$\begin{aligned} (3\alpha + 2)K_{h1} + K_{h2}\alpha &= 0 \\ K_{h1}\alpha + (4\alpha + 2)K_{h2} &= 0 \end{aligned} \tag{L.41}$$

We can combine the two equations in (L.41) to form,

$$\frac{\alpha}{3\alpha + 2} = -\frac{K_{h1}}{K_{h2}} = \frac{4\alpha + 2}{\alpha} \tag{L.42}$$

or $11\alpha^2 + 14\alpha + 4 = 0$. Therefore the two eigenvalues are $\alpha_1 = -0.4331$, $\alpha_2 = -0.8396$.. The forms of the homogeneous solutions are $\mathrm{y}_{1h}[n] = K_{h11}\alpha_1^n + K_{h12}\alpha_2^n$ and $\mathrm{y}_{2h}[n] = K_{h21}\alpha_1^n + K_{h22}\alpha_2^n$ and, enforcing(L.42),

$$\frac{K_{h11}}{K_{h21}} = 0.618 \quad \text{and} \quad \frac{K_{h12}}{K_{h22}} = -1.618\ . \tag{L.43}$$

Since the forcing functions are constants we can assume particular solutions $\mathrm{y}_{1p}[n] = K_{p1}$ and $\mathrm{y}_{2p}[n] = K_{p2}$. Doing that and solving, we get $K_{p1} = -0.1724$ and $K_{p2} = 0.8621$. So the total solution is of the form

$$\mathrm{y}_1[n] = K_{h11}\alpha_1^n + K_{h12}\alpha_2^n - 0.1724 \ \text{ and }\ \mathrm{y}_2[n] = K_{h21}\alpha_1^n + K_{h22}\alpha_2^n + 0.8621$$

We can now use $\mathrm{y}_1[0] = 0$ and $\mathrm{y}_2[0] = 2$. Applying the initial conditions we get

$$\mathrm{y}_1[0] = K_{h11} + K_{h12} - 0.1724 = 0 \quad \text{and} \quad \mathrm{y}_2[0] = K_{h21} + K_{h22} + 0.8621 = 2\,.$$

These two equations, together with (L.43) lead to the final numerical solution

$$\begin{aligned} K_{h11} = 0.557\ \ &,\ K_{h12} = -0.3841 \\ K_{h21} = 0.9005\ \ &,\ K_{h22} = 0.2374 \end{aligned}$$

$$\mathrm{y}_1[n] = 0.557(-0.4331)^n - 0.3841(-0.8396)^n - 0.1724 \tag{L.44}$$

and

$$y_2[n] = 0.9005(-0.4331)^n + 0.2374(-0.8396)^n + 0.8621 \; . \qquad (L.45)$$

Using vectors and matrices there is a more systematic way of solving systems of differential and difference equations. (See Appendix J.)

Exercises

(On each exercise, the answers listed are in random order.)

1. Find the solutions of these differential equations with the boundary conditions indicated.

   (a) $y' = -10y$, $y(0) = 1$ (b) $3y' - 4y = 0$, $y(2) = -1$

   (c) $\frac{y'}{2} + y = 0$, $\frac{d}{dt}(y(t))\Big|_{t=0} = 4$

   Answers: $y(t) = -0.069e^{4t/3}$ $y(t) = e^{-10t}$ $y(t) = -2e^{-2t}$

2. Find the solutions of these differential equations with the boundary conditions indicated.

   (a) $y' + 10y = 5$, $y(0) = 0$

   (b) $3y' - 4y = 10\cos(20\pi t)$, $y(0) = 0$

   (c) $y'' + 10y' + 100y = e^{-5t}$, $y(0) = 10$, $\frac{d}{dt}(y(t))\Big|_{t=0} = -1$

   (d) $4y'' + 10y' + 8y = \sin(10\pi t)$, $y(0) = 0$, $\frac{d}{dt}(y(t))\Big|_{t=0} = 0$

   (e) $y''(t) + 5y'(t) + 10y(t) = 4$, $y(0) = 1$, $\frac{d}{dt}(y(t))\Big|_{t=0} = -3$.

   Answers:

$$y(t) = \left(10.05\times10^{-6} - j0.003013\right)e^{-(5+j\sqrt{7})t/4} + \left(10.05\times10^{-6} + j0.003013\right)e^{-(5-j\sqrt{7})t/4}$$
$$- 0.2522\times10^{-3}\sin(10\pi t) - 20.1\times10^{-6}\cos(10\pi t)$$

$$y(t) = \frac{1}{2}\left(1 - e^{-10t}\right)$$

$$y(t) = (4.993 - j2.83)e^{-(10-j\sqrt{300})t/2} + (4.993 + j2.83)e^{-(10+j\sqrt{300})t/2} + \frac{e^{-5t}}{75}$$

$$y(t) = 0.00113e^{4t/3} - 0.00113\cos(20\pi t) + 0.053\sin(20\pi t)$$

$$y(t) = e^{-5t/2}\left[0.6\cos\left(\sqrt{15}t/2\right) - \frac{3}{\sqrt{15}}\sin\left(\sqrt{15}t/2\right)\right] + 0.4$$

3. Solve the system of differential equations

$$y_1' + 2y_1 + 8y_2 = 0$$
$$y_2' + y_2 + 5y_1 = -4$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = -6$.

Answer:

$$y_1(t) = -1.844e^{-7.844t} + 5.686e^{4.844t} - 0.8421$$

$$y_2(t) = -1.347e^{-7.844t} - 4.864e^{4.844t} + 0.2105$$

4. Find the first derivative and then approximations to the first derivative, using the first forward, backward and central differences, of the function $x(t) = e^{-t}$ at time $t = 1$, using $\Delta t = 1, 0.1$ and 0.01.

Answers: –0.3697 –0.3684 –0.4323 –0.36788 –0.366
–0.35 –0.387 –0.36788 –0.632 –0.2325

5. Convert the differential equation with an initial condition

$$4y'(t) + 8y(t) = 0 \ , \ y(0) = -10$$

to a difference equation using backward differences, with $\Delta t = 0.05$, solve the resulting difference equation and plot a graph of the discrete-time solution $y[n]$ vs. n for $0 \le n < 40$. Compare the solution to samples of the solution of the differential equation $y(t) = -10e^{-2t}$.

6. Repeat Exercise 5 using forward differences instead of backward differences and compare the solutions.

7. Find the total solution to the difference equation $4y[n+1] + y[n] = 0$ with the initial condition $y[0] = -5$.

Answer $y[n] = -5(-1/4)^n$

8. Find the total solution to the difference equation $y[n] + y[n-1] + 2y[n-2] = 0$ with the initial conditions $y[0] = 5$ and $y[1] = 3$.

Answer: $y[n] = 6.5(1.414)^n \cos(-0.694 + 1.932n)$

9. A discrete-time function $y[n]$ obeys the difference equation $2y[n] + y[n-1] = 10$ and satisfies the initial condition $y[0] = 4$. Using the solution complete this table:

n	$y[n]$
1	__
2	__
3	__
4	__

Check your total solution by testing the numbers in the table against the original difference equation.

10. Find the total solution of the difference equation $y[n]-4y[n-1]=e^{-2n}$ subject to the initial conditions $y[0]=0$.

Answer $\quad y[n]=0.035\left[(4)^n-e^{-2n}\right]$

11. Find the total solution of the difference equation $y[n]+10y[n-1]=8$ subject to the initial condition $y[0]=2$.

Answer $\quad y[n]=\dfrac{14(-10)^n+8}{11}$

12. Find the total solution of the difference equation $3y[n]+8y[n-1]+4y[n-2]=2\cos\left(\dfrac{2\pi n}{7}\right)$ subject to the initial conditions $y[0]=-1$ and $y[1]=0$.

Answer

$y[n]=-2.237(-2/3)^n+0.903(-2)^n+0.1355\sin(2\pi n/7)+0.334\cos(2\pi n/7)$

13. Find the total solution of the difference equation $y[n]+4y[n-1]+2y[n-2]=\cos(n/8)$ subject to the initial conditions $y[0]=3$, $y[1]=0$.

Answer:

$y[n]=3.385\left(-2+\sqrt{2}\right)^n-0.523\left(-2-\sqrt{2}\right)^n+0.1651\cos(n/8)+0.1412\sin(n/8)$

14. Solve the two-difference-equation system

$$5y_1[n]+2y_1[n-1]-y_2[n-1]=2$$
$$-6y_2[n]+3y_2[n-1]+3y_1[n-1]=0$$

with initial conditions $y_1[0]=-4$ and $y_2[0]=1$.

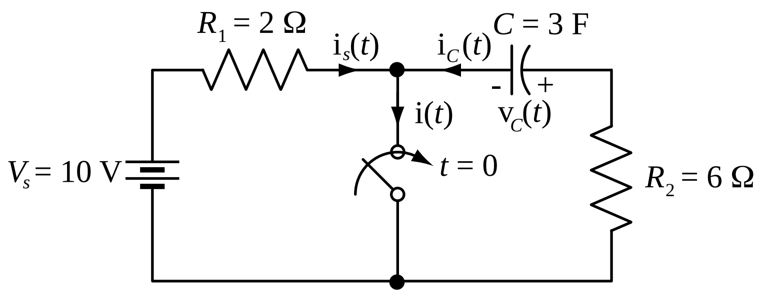
Answer:

$$y_1[n] = \frac{45}{54}\alpha_1^n - \frac{279}{54}\alpha_2^n + \frac{1}{3} \quad \text{and} \quad y_2[n] = \frac{5}{54}\alpha_1^n + \frac{31}{54}\alpha_2^n + \frac{1}{3}.$$

15. After time $t = 0$ a circuit is described by the differential equation $R\mathrm{i}(t) + L\mathrm{i}'(t) = A$ where $\mathrm{i}(t)$ is the current through the series combination of a resistor R an inductor L and a voltage source and the voltage source is a constant voltage A. Assume that there is initially no stored energy in the circuit. Then the initial current must be zero because of the inductor. That is, $\mathrm{i}(0^+) = 0$. If the resistance R is 10 Ω and the inductance L is 2 H and A is 10 V, find the total numerical solution $\mathrm{i}(t)$ for all time.

16. A current source $5\sin(20000\pi t)$ is suddenly applied at time $t = 0$, to the parallel combination of an inductance of 10 mH, a resistance of 20 Ω and a capacitance of 5 µF. There is no energy stored in the circuit before time $t = 0$. Find the total numerical solution for the voltage across the four parallel elements for all time.

17. Write and solve a differential equation for the voltage $\mathrm{v}_C(t)$ in the circuit below for time $t > 0$ then find an expression for the current $\mathrm{i}(t)$ for time $t > 0$.



18. (a) Find the first derivative and then approximations to the first derivative, using the first forward, backward and central differences, of the function $\mathrm{x}(t) = \cos(8\pi t)$ at time $t = 1/32$ using $\Delta t = 0.25, 0.1, 0.01$.

(b) Approximate the differential equation for the voltage $\mathrm{v}_C(t)$ found in Exercise 17 with a difference equation, using a Δt of one-tenth of the circuit's time constant and numerically solve for the voltage $\mathrm{v}_C(t)$ over a time span of 5 time constants beginning at time $t = 0$ by iteration using MATLAB. Graph the exact solution $\mathrm{v}_C(t)$ and the numerical solution $\mathrm{v}_C[n\Delta t]$ on the same time scale for comparison.

19. A second derivative is a first derivative of a first derivative. Then a second difference should be a first difference of a first difference. Show that a second difference can be written as

$$\Delta^2 \mathrm{x}[n] = \Delta(\Delta \mathrm{x}[n]) = \mathrm{x}[n+2] - 2\mathrm{x}[n+1] + \mathrm{x}[n]$$

and that a third difference can be written as

$$\Delta^3 \mathrm{x}[n] = \Delta(\Delta(\Delta \mathrm{x}[n])) = \mathrm{x}[n+3] - 3\mathrm{x}[n+2] + 3\mathrm{x}[n+1] - \mathrm{x}[n] \quad .$$

20. Mortgage loan payments are usually based on a system of payment in which the debt is retired by making equal monthly payments over the life of the loan. Every month interest accumulates on the unpaid balance. In each monthly payment some of it pays interest costs and the rest is applied to the principle. Assign these variables:

A	Loan amount
r	Annual interest rate in percent
M	Total payment at the end of the nth month, a constant
$I[n]$	Interest payment at the end of the nth month
$P[n]$	Principle payment at the end of the nth month
N	Total number of months to pay the loan

Write difference equations expressing the relationships between these quantities. Let the time increment be one month. Find formulas for the monthly interest payment, the monthly principle payment and the overall monthly payment. Then find the monthly payment for a \$100,000 loan for 30 years at a 10% annual interest rate. (The formula for the summation of a finite geometric series

$$\sum_{n=0}^{N-1} r^n = \begin{cases} N & , \quad r = 1 \\ \dfrac{1-r^N}{1-r} & , \quad r \neq 1 \end{cases}$$

may prove useful.)

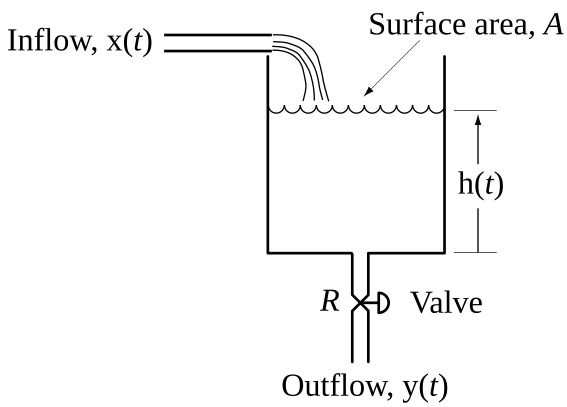
21. A water tank is filled by an inflow $\mathrm{x}(t)$ and is emptied by an outflow $\mathrm{y}(t)$. The outflow is controlled by a valve which offers resistance R to the flow of water out of the tank. The water height in the tank is $\mathrm{h}(t)$ and the surface area of the water is A, independent of height (cylindrical tank). The outflow is related to the water height (head) by

$$\mathrm{y}(t) = \frac{\mathrm{h}(t)}{R}.$$

(a) Write the differential equation for the water height.

(b) If the valve resistance is $10\ \mathrm{s/m^2}$, and the inflow is $0.05\ \mathrm{m^3}/s$, at what water height will the inflow and outflow rates be equal, making the water height constant?

(c) Let the tank's full height be 1.5 m, let the initial water height at time $t = 0$ be 0.5 m and let the tank diameter be 1 m. If the inflow is a constant $0.2\ \mathrm{m^3}/s$ when will the tank start to overflow?



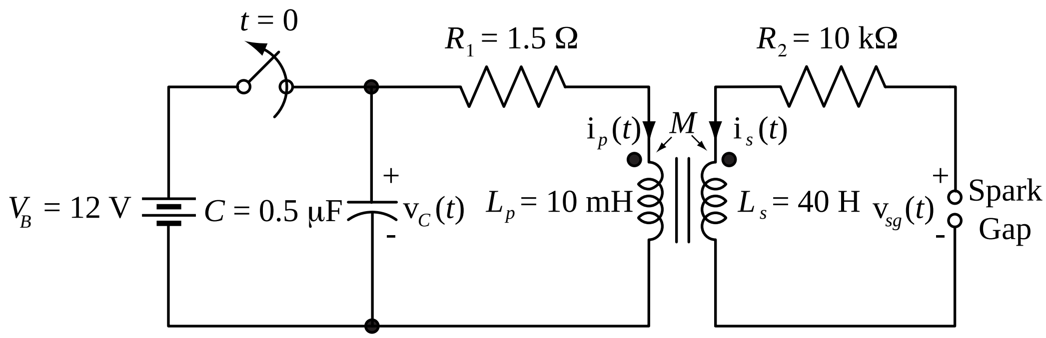
22. Pharmacokinetics is the study of how drugs are absorbed into, distributed through, metabolized by and excreted from the human body. Some drug processes can be approximately modeled by a "one compartment" model of the body in which V is the volume of the compartment, C_p is the drug concentration in that compartment, k_{el} is a rate constant for excretion of the drug from the compartment and k_0 is the infusion rate at which the drug enters the compartment. These can be combined into a single differential equation for the drug concentration as a function of time,

$$V\frac{d}{dt}\left(C_p(t)\right) = k_0 - Vk_{el}C_p(t) \ .$$

(a) Let the parameter values be $k_{el} = 0.4\,\text{hr}^{-1}$, $V = 20\ \text{l}$ and $k_0 = 100$ mg/hr (where "l" is the symbol for liter). If the initial drug concentration is $C_p(0) = 10\ \text{mg/l}$, plot the drug concentration as a function of time for 10 hrs.

23. The decay of radioactive substances is governed by the principle that the rate of decay is proportional to the amount of the substance remaining. If the decay rate constant, k, of radium is $-1.4\times10^{-11}\,\text{s}^{-1}$ what is its half-life in years? (Half-life is the time in which half of the original amount of a substance has decayed.)

24. Automobile ignition operates on the general principle of storing energy in an inductor (the spark coil) and then releasing the energy into the spark plug in such a way as to produce a high voltage across the spark gap, causing it to arc and thereby ignite the fuel-air mixture in the combustion chamber. The circuit including the spark coil (transformer) and condensor is first connected to a voltage source through a small resistance so that a current builds up in the primary winding of the coil. Then the circuit is disconnected from the voltage source. Up to the time the spark gap arcs the circuit can be analyzed by linear circuit analysis. Let the circuit be represented by the model below. Write and solve a system of two differential equations for the capacitor voltage and the transformer primary current. If the spark gap arcs at 50 kV, how long after the switching occurs does the arc occur? (Assume the transformer windings are completely coupled, $M = \sqrt{L_p L_s}$.)



25. An aluminum block is heated to a temperature of 100 °C. It is then dropped into a flowing stream of water which is held at a constant temperature of 10°C. After 1 minute the temperature of the block is 60°C. (Aluminum is such a good heat conductor that its temperature is essentially uniform throughout its volume during the cooling process.) Assuming that the rate of cooling is proportional to the temperature difference between the block and the water, find the time required for the block to reach a temperature of 15°C.

26. A well-stirred vat has been fed for a long time by two streams of liquid, fresh water at 0.2 cubic meters per second and concentrated blue dye at 0.1 cubic meter per second. The vat contains 10 cubic meters of this mixture and the mixture is being drawn from the vat at a rate of 0.3 cubic meters per second to maintain the volume. The blue dye is suddenly changed to red dye at the same flow rate. At what time after the switch does the mixture drawn from the vat contain a ratio of red to blue dye of 99:1?